Rolling Manifolds of Arbitrary Dimensions

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Résumé

Nous étudions dans cette thèse le roulement sans glissement et sans pivotement de deux variétés lisses M et $\hat{M}$ l'une sur l'autre de dimensions n et $\hat{n}$ respectivement. L'objectif principal est de chercher des conditions nécessaires et suffisantes de la commandabilité du système commandé défini par le roulement.

Dans le premier chapitre, on présente les motivations et le plan de la thèse ainsi les notations utilisées le long des chapitres.

Dans le deuxième chapitre, on caractérise l'espace d'état du roulement quand M et $\hat{M}$ sont des variétés Riemanniennes lorsque n n'est pas nécessairement égal à $\hat{n}$ et du développement quand M et $\hat{M}$ sont des variétés affines munies des connexions affines avec $n = \hat{n}$. Ainsi, on donne les relèvements et les distributions correspondant aux deux notions précédentes.

Le troisième chapitre contient quelques résultats de la commandabilité du système de roulement des variétés Riemanniennes. Plus précisément, on présente les conditions nécessaires de la non-commandabilité du roulement d'une variété Riemannienne 3-dimensionnelle sur une autre 2-dimensionnelle.

Le chapitre 4 porte sur le roulement d'une variété Riemannienne de dimension 2 sur une autre de dimension 3. On trouve que la dimension d'une orbite non-ouverte quelconque de l'espace d'état appartient à $\{2, 5, 6, 7\}$. Les aspects géométriques de deux variétés sont liés principalement avec le fait que la variété de dimension 3 contient une sous-variété totalement géodésique de dimension 2.

Dans le dernier chapitre, on introduit et étudie un concept d'holonomie horizontale associé à un triplet (M, ∇, Δ) avec M variété différentielle connexe, ∇ connection affine complète sur M et Δ distribution complètement commandable. Si H^{∇} est le groupe d'holonomie associé à (M, ∇), on considère alors son sous-groupe obtenu uniquement en considérant le transport ∇-parallèle par rapport aux lacets dans M tangents à la distribution Δ. On le note H^{∇}_{Δ} et on l'appèle groupe d'holonomie horizontal. On prouve que le groupe d'holonomie horizontal H^{∇}_{Δ} est un sous-groupe de Lie de $\mathrm{GL}(n)$. Malgré que la distribution considérée est complètement commandable, on a démontré par un exemple que l'inclusion du groupe d'holonomie horizontale est peut-être stricte dans le groupe d'holonomie. A cette fin, on utilise le modèle du roulement avec M un groupe de Carnot homogène munie d'une connexion de Levi-Civita associée à une métrique Riemannienne sur l'espace Euclidien $\mathbb{R}^n$ munie de la connexion Euclidienne.

Mots-clefs: Variétés roulantes, développement des variétés, commandabilité, géométrie Riemannienne, espace fibré, théorie des groupes de Lie, groupe d'holonomie.

Abstarct

In this thesis, we study the rolling motion without spinning nor slipping of a smooth manifolds M and $\hat{M}$ against another of dimensions n and $\hat{n}$ respectively. The purpose

is to find the necessary and sufficient conditions for the controllability issue of the system of rolling. We start by a french review of the principal results of the thesis is included in the introduction.

In Chapter 1, we present the motivations of the subject thesis, the structure of the contents and the notations used along the manuscript.

The second chapter contain a characterization of the state space of rolling manifolds when M and $\hat{M}$ are Riemannian manifolds with n and $\hat{n}$ are not necessarly equal and of the development of manifolds when M and $\hat{M}$ are affine manifolds of dimension $n = \hat{n}$ equipped with affine connections. We also state the definitions of the lifts and the distributions with respect to the previous notions.

The controllability results of the rolling system of Riemannian manifolds is included in Chapter 3. We give all the necessary conditions of the non-controllability of rolling of 3-dimensional Riemannian manifold against 2-dimensional Riemannian manifold.

Chapter 4 delas with the rolling of a 2-dimensional Riemannian manifold against a 3-dimensional Riemannian manifold. We prove that the dimension of an arbitrary non-open orbit of the state space belongs to $\{2, 5, 6, 7\}$. The geometrical aspects of the two manifolds depend on the existence of a 2-dimensional totally geodesic submanifold in the 3-dimensional manifold.

The last chapter introduces and addresses the issue of horizontal holonomy associated to a triple (M, ∇, Δ) with M smooth connected manifold, ∇ complete affine connection on M and Δ completely controlable distribution over M. If H^{∇} denotes the holonomy group associated with (M, ∇), one considers its subgroup obtained by considering only the ∇-parallel transport with respect to loops of M tangent to the distribution Δ. This subgroup is denoted by H^{∇}_{Δ} and we call it horizontal holonomy group. We prove that the horizontal holonomy group H^{∇}_{Δ} is a Lie subgroup of $\mathrm{GL}(n)$. Despite that the considered distribution is completely controllable, we show by means of an explicit example that the inclusion of such horizontal holonomy group can be strict in the holonomy group. To this end, we use the rolling problem of M taken as a step 2 homogeneous Carnot group equipped with the Levi-Civita connection associated to a Riemannian metric onto the Euclidean space $\mathbb{R}^n$ equipped with the Euclidean connection.

Keywords: Rolling manifolds, development of manifolds, controllability, Riemannian geometry, fiber bundle, theory of Lie groups, holonomy groups.

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Contents

Chapter 1

Un Aperçu de la Thèse (Thesis Summary in French)

1.1 Introduction

Ces dernières années, les études sur les systèmes dynamiques commandés de différentes disciplines se progressent très vivement. Parmi ces systèmes, nous nous intéressons dans cette thèse par le modèle de roulement de deux variétés Riemanniennes connexes et orientés (M, g) et $(\hat{M}, \hat{g})$ de dimensions n et $\hat{n}$ respectivement. Deux types de contraintes sont considérés, ils sont notés par roulement sans pivotement et roulement sans pivotement et sans glissement. La plupart des travaux dans ce domaine traite le cas où les deux variétés ont même dimensions [8, 9, 11].

En particulier, considérons le problème de roulement de deux surfaces convexes l'une sur l'autre comme par exemple le roulement d'une boule sur un plan dans $\mathbb{R}^3$. Lorsque les deux surfaces sont en point de contact, leurs vecteurs normaux extérieurs sont opposés en ce point. Un tel roulement sans glissement exige qu'au point de contact, le vecteur vitesse tangent à la première surface est égale au vecteur vitesse tangent à la deuxième surface tourné d'un angle $\theta(\cdot)$. Tandis que la condition d'absence de pivotement signifie que les axes de rotation de deux surfaces restent dans le plan tangent commun, ce qui traduit en une condition sur $\dot{\theta}(\cdot)$. Alors, l'espace d'état de ce modèle de roulement sans pivotement, ni glissement est de dimension cinq parce que, comme on a vu, un point de cet espace est défini en fixant un point sur chaque surface et un angle.

Généralement, l'espace d'état de roulement des deux variétés Riemanniennes est un espace fibré et sa fibre typique est l'ensemble des isometries formées entre les espace tangents des variétés considérées. Géométriquement, le roulement sans rotation signifie que l'image d'un champ de vecteurs parallèle à une courbe dans M par A est un champ de vecteurs parallèle à une courbe dans $\hat{M}$. Comme ceci, la distribution de roulement sans pivotement $\mathscr{D}_{NS}$ sur Q est définie comme étant la dérivation du transport parallèle des champs de vecteurs X et $\hat{X}$ tangents à M et $\hat{M}$ respectivement. Par ailleurs, la distribution $\mathscr{D}_R$ qui décrive les deux containtes du roulement est une sous-distribution de $\mathscr{D}_{NS}$, obtenue en faisant $\hat{X} = AX$. Ainsi, on a défini les deux systèmes dynamiques commandés affines sans dérives $(\Sigma)_{NS}$ et $(\Sigma)_R$ sur Q correspondants à $\mathscr{D}_{NS}$ et à $\mathscr{D}_R$ respectivement.

Le point focal est de résoudre le problème de la commandabilité de $(\Sigma)_R$ et $(\Sigma)_{NS}$ par des outils géométriques sur M et $\hat{M}$. Plus précisement, c'est de chercher des conditions nécessaires et/ou des conditions suffisantes afin d'obtenir que pour toute paire (q_{init}, q_{final}) de points dans Q, il existe une courbe $q(\cdot)$ tangente à $\mathscr{D}_R$ (à $\mathscr{D}_{NS}$ respectivement) et qui oriente le système $(\Sigma)_R$ ($(\Sigma)_{NS}$ respectivement) de (q_{init} à q_{final}). Cette approche nous motive de chercher l'ensemble atteignable par de telles courbes à partir d'un point quelconque $q_0 \in Q$. Ce sont les orbites de roulement associées à $\mathscr{D}_{NS}$ et $\mathscr{D}_R$. On dit que le système de roulement est complètement commandable si les orbites de roulement des certains points sont égales à l'espace d'état.

Au titre de l'exemple précédent, c'est-a-dire lorsque M et $\hat{M}$ sont deux variétés Riemanniennes de dimensions 2, le système $(\Sigma)_R$ est complètement commandable si et seulement si les variétés ne sont pas isométrique. En particulier, la dimension des ensembles atteignables soit 2 ou 5 (cf. [1]). D'autre part, [11] nous donne des réponses satisfaisantes pour la question de la commandabilité de $(\Sigma)_R$ dans le cas où les deux variétés sont de dimensions 3. Les mêmes auteurs fournissent des conditions nécessaires et suffisantes pour la commandabilité lorsque l'une des variétés est de courbure constante (cf. [12]).

Les travaux de Y. Chitour et P. Kokkonen traitent le cas de roulement sans pivotement et sans glissement d'une variété Riemannienne, lisse, connexe et orientée (M, g) sur une variété Riemannienne complète $(\hat{M}, \hat{g})$ de courbure constante et de même dimension que M. En particulier, si la courbure de $\hat{M}$ est nulle, alors $(\Sigma)_R$ est complètement controllable si et seulement si le groupe d'holonomie de M, par rapport à la connexion de Levi-Civita, est égal au groupe spécial orthogonal $\mathrm{SO}(n)$. Le deuxième auteur explique dans [24] et [26] le développement des variétés affines qui n'ont pas nécessairement des tenseurs de torsion nuls.

Dans Chapitre 2 de cette thèse, on présente dans deux parties les définitions d'espace d'état, des relèvements, des distributions de roulement des deux variétés Riemanniennes, lisses, connexes et du développement de roulement des variétés affines. La référence de base dans la première partie est [10] où on étend ses définitions et ses résultats vers le cas où les deux dimensions n et $\hat{n}$, des variétés considérées (M, g) et $(\hat{M}, \hat{g})$ respectivement, ne sont pas égales. Par exemple, lorsque $n \geq \hat{n}$, il nous faut prendre en compte que l'image d'un isométrie A doit être dans l'espace tangent $T_{\hat{x}}\hat{M}$ en un point fixé $\hat{x}$. Dans la deuxième partie, on joint le contexte de développement des variétés affines qui est défini dans [24] et le contexte de roulement sans pivotement par rapport à une connexion affine quelconque qui est donné dans [26], alors on définit le développement de roulement des variétés affines.

On introduit les principaux résultats sur la controllabilité du système de roulement des variétés Riemanniennes de différentes dimensions dans Chapitre 3. Concernant le système $(\Sigma)_{NS}$, dû aux études dans [8] et [9], la résolution du problème de la commandabilité est clairement réliée à la caractérisation des groupes d'holonomie de M et $\hat{M}$. Alors que dans le cas de $(\Sigma)_R$, la situation est plus compliquée. La première propriété remarquée est que la distribution de roulement $\mathscr{D}_R$ est de dimension n. Ce qui nous conduit à la conclusion suivante: le problème de roulement des deux variétés à deux dimensions différentes n'est pas symétrique en prenant en compte l'ordre de deux variétés. D'où, la stratégie consiste à calculer les crochets de Lie dans l'espace des sections de $\mathscr{D}_R$. Le premier crochet de Lie duquels nous permet de définir le tenseur Rol disant *la courbure de roulement*, il peut-être considéré comme étant la différence entre les deux tenseurs de courbure de Riemann de M et $\hat{M}$. On constate que la commandabilité de $(\Sigma)_R$ se rattache d'une manière directe à ces valeurs des courbures et à leurs dérivées covariantes. Dans cette description, il y en a un résultat important: lorsque $n < \hat{n}$ strictement et la courbure de $\hat{M}$ est constante alors le problème de roulement sans pivotement et sans glissement n'est pas commandable.

Par contre, lorsque $(n, \hat{n}) = (2, 3)$, on a réussi à trouver quelques conditions nécessaires pour la non-commandabilité dont soit $\mathscr{D}_R$ est involutive et M est isométrique à une sous-variété totalement géodésique de $\hat{M}$, soit $\hat{M}$ est un produit tordu d'un intervalle réel avec une sous-variété totalement géodésique de dimension 2, soit M est de courbure constante et $\hat{M}$ est isométrique localement au produit Riemannien d'un intervalle réel avec une sous-variété de dimension 2. Les calculs de Chapitre 4 montrent aussi que la dimension de chaque orbite appartient à l'ensemble $\{2, 5, 6, 7\}$.

Dans le dernier chapitre, on définit le groupe d'holonomie H^{∇}_{Δ} à partir des lacets tangents à une distribution complètement commandable Δ sur une variété affine M par rapport à une connexion affine ∇. De plus, le développement de roulement d'une variété affine (M, ∇) sur $(\mathbb{R}^n, \hat{\nabla}^n)$, où $\hat{\nabla}^n$ est la connexion Euclidienne, nous permet de définir le groupe d'holonomie affine $\mathcal{H}^{\nabla}$ sur M et le groupe d'holonomie horizontal affine $\mathcal{H}^{\nabla}_{\Delta_R}$ sur Δ. Ce sont des sous-groupes du groupe $\mathrm{Aff}M$ de tous les transformations affines inversibles de M sur elle-même. Nous avons prouvé que $\mathcal{H}^{\nabla}_{\Delta_R}$ est un sous-groupe de Lie de $\mathrm{Aff}(n)$ et H^{∇}_{Δ} est un sous-groupe de Lie de $\mathrm{GL}(n)$. On a aussi répondu à une question n'est pas évidente telqu'on a montré qu'en général, la fermeture de groupe d'holonomie horizontal restreint $(H^{\nabla}_{\Delta})^0$ n'est pas égale au groupe d'holonomie complet de M. Pour cela, on a étudié le roulement d'un groupe de Carnot homogène d'ordre 2 (M, g) munie de la métrique Riemannienne g sur $(\mathbb{R}^n, s^n)$ où s^n est la métrique Euclidienne. On a le résultat grâce au fait que les groupes d'holonomie affines ont les même structures géométriques des orbites de développement de roulement sur un espace Euclidien.

1.2 Notations

Le long de cette thèse, on note par A^i_j l'élément en $i^{\text{ème}}$-ligne et $j^{\text{ème}}$-colonne d'une matrice réelle A et par A^T sa matrice transposée. En outre, soit $L : V \to W$ un morphisme $\mathbb{R}$-linéaire où V et W sont deux espaces $\mathbb{R}$-linéaires de dimensions n et n' respectivement. Si $F = (v_i)_{i=1}^n$ et $G = (w_i)_{i=1}^{n'}$ sont deux bases de V et W respectivement, alors la $(n' \times n)$-matrice réelle qui représente L par rapport aux F et G est notée par $\mathcal{M}_{F,G}(L)$ et donnée par $L(v_i) = \sum_j \mathcal{M}_{F,G}(L)^j_i w_j$. Par ailleurs, si g et h sont des produits scalaires sur V et W respectivement, nous symbolisons l'application transposé (adjointe) de L par rapport à g et h par $L^{T_{g,h}} : W \to V$ où $g(L^{T_{g,h}}w, v) = h(w, Lv)$. Ainsi, on écrit $(\mathcal{M}_{F,G}(L))^T = \mathcal{M}_{F,G}(L^{T_{g,h}})$.

Dans la suite, toutes les variétés considérées sont lisses, connexes et de dimension finie. Soient E, M, F des variétés différentielles, un espace fibré $\pi_{E,M} : E \to M$ est un morphisme lisse qui vérifie que pour tout $x \in M$ il existe un voisinage U de x dans M et un difféomorphisme $\tau : \pi^{-1}_{E,M}(U) \to U \times F$ tel que $pr_1 \circ \tau = \pi_{E,M}|^{-1}_{\pi(U)}$, où pr_1 signifie la projection sur le premier facteur. On appelle $E|_x := \pi^{-1}_{E,M}(x)$ la fibre au-dessus du point x, F la fibre typique de $\pi_{E,M}$ et τ sa trivialisation locale. La fibre typique est unique à un difféomorphisme près. De plus, une section lisse du fibré $\pi_{E,M}$ est un lisse morphisme $s : M \to E$ qui satisfait $\pi_{E,M} \circ s = \mathrm{id}_M$. L'ensemble de toutes les sections de π est noté par $\Gamma(E)$. Si F est un espace $\mathbb{R}$-linéaire de dimension finie, on dit que $\pi_{E,M}$ est un fibré vectoriel. D'autre part, si G est un groupe de Lie, alors $\pi : E \to M$ est un fibré principal de groupe structural G sur M s'il existe une action libre de G sur E qui conserve les fibres de π (cf. [23]).

Soit $\Gamma(E)$ l'espace des sections lisses du fibré vectoriel E sur M. Une connexion linéaire sur E est un morphisme $\Gamma(E) \to \Gamma(E \times T^\star M)$ linéaire sur $\mathbb{R}$ et qui vérifie la règle de Leibniz sur l'ensemble des fonctions lisses $C^\infty(M)$. On note par $\mathcal{X}(M)$ l'ensemble des sections lisses du fibré tangent TM. Une connexion affine ∇ sur M est une connexion linéaire sur TM et qui est donnée par le morphisme bilinéaire sur $\mathbb{R}$ suivant,

$$\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M); \quad (X, Y) \mapsto \nabla_X Y.$$

Elle est linéaire sur $C^\infty(M)$ par rapport au premier facteur et vérifiée la règle de Leibniz sur $C^\infty(M)$ par rapport au deuxième facteur. Ainsi, on dit que (M, ∇) est une variété affine. De plus, si l'application exponentielle exp^∇_x de (M, ∇) est définie sur l'ensemble de l'espace tangent T_xM pour tout $x \in M$, alors (M, ∇) est géodésiquement complète. La connexion Euclidienne ∇^n sur $\mathbb{R}^n$ est une connexion affine tel que $\nabla^n_{E_i} E_j = 0$ pour n'importe quels deux champs de vecteurs entre la base canonique $\{E_1, \dots, E_n\}$ sur $\mathbb{R}^n$. Les définitions intrinsèques du tenseur de courbure R^∇ et du tenseur de torsion T^∇ associés à la connexion affine ∇ sont,

$$\begin{aligned} R^\nabla(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ T^\nabla(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y], \end{aligned}$$

où X, Y et Z sont des champs de vecteurs quelconques sur M. Si $T^\nabla(X, Y)$ est nulle pour tout champs de vecteurs X et Y sur M, alors on dit que la connexion ∇ est sans torsion ou symétrique. Une variété M munie d'une métrique définie positive g est appelée variété Riemannienne et notée par (M, g). De plus, dans cette thèse, on admet qu'une variété Riemannienne est complète et orientée. Dans ce dernier cas, on utilise $\|v\|_g$ pour noter $g(v, v)^{1/2}$ pour tout $v \in T_xM$ et $x \in M$.

On rappelle qu'une distribution Δ sur la variété M est un sous-fibré lisse de TM de telle sorte que pour tout $x \in M$, $\Delta|_x$ est localement engendrée par m champs de vecteurs linéairement indépendants. On dit alors que Δ est de dimension m. A partir de $\Delta|_x$, on définit, en $x \in M$, la collection des distributions $\Delta^1|_x \subset \Delta^2|_x \subset \cdots \subset \Delta^r|_x \subset \dots$ vérifiée $\Delta^1|_x := \Delta|_x$ et $\Delta^{s+1}|_x := \Delta^s|_x + [\Delta^1, \Delta^s]|_x$ pour $s \geq 1$. On dit que la distribution Δ est complètement commandable si, pour tout $x \in M$, il exist un entier $r = r(x)$ tel que $\Delta^r|_x = T_xM$. Ainsi, r est l'ordre de $\Delta|_x$. D'autre part, une courbe absolument continue $c : I \to M$ définie sur un intervalle réel $I \subset \mathbb{R}$ est Δ-admissible courbe si elle est tangente à Δ presque partout (p.p.): $\dot{c}(t) \in \Delta|_{c(t)}$ pour presque tout $t \in I$. Fixons $x_0 \in M$, une Δ-orbite en x_0 est l'ensemble de points finaux atteints par toutes les courbes Δ-admissibles qui commencent par x_0. Par Théorème de l'Orbite, $\mathcal{O}_\Delta(x_0)$ est une sous-variété lisse immergée de M contenant x_0 (cf. [1], [20]). Il convient de noter qu'on peut restreindre la définition de l'orbite aux courbes continues par morceaux. Une distribution Δ' sur M est une sous-distribution de Δ si $\Delta' \subset \Delta$. Evidemment, on a $\mathcal{O}_{\Delta'}(x_0) \subset \mathcal{O}_\Delta(x_0)$ pour tout $x_0 \in M$. On note que l'évaluation de l'algèbre de Lie engendré par Δ en $x \in M$ par $Lie(\Delta)_x$.

Soit $\pi : E \to M$ un espace fibré et $y \in E$. On note par $V\mid_y(\pi)$ l'ensemble de tous les champs de vecteurs $Y \in T\mid_y E$ tels que $\pi_*(Y) = 0$. Si π est un espace fibré alors la collection des espaces $V|_y(\pi)$, $y \in E$ définit une sous-variété lisse $V(\pi)$ de TE, c'est la distribution verticale. Dans ce dernier cas, $\pi_{V(\pi)} := \pi_{TE}|_{V(\pi)}$ est un sous-fibré vectoriel de $\pi_{TE} : TE \to E$.

Supposons maintenant que $\pi : E \to M$ et $\eta : F \to M$ sont deux fibrés vectoriels sur la variété différentielle M. On note par $C^\infty(\pi, \eta)$ l'ensemble des morphismes $f : E \to F$ tel que $\eta \circ f = \pi$. Fixons $f \in C^\infty(\pi, \eta)$ et $u, w \in \pi^{-1}(x)$, alors la dérivée verticale de f en u dans la direction de w est le morphisme $\mathbb{R}$-linéaire suivant

$$\begin{array}{rcl} \nu(.)|_u(f) : \pi^{-1}(x) & \to & \eta^{-1}(x) \\ w & \mapsto & \nu(w)\mid_u(f) := \frac{d}{dt}|_0 f(u + tw). \end{array} \tag{1.1}$$

Selon cette définition, $\nu(w)|_u$ est bien un élément de $V|_u(\pi)$. En conséquence, $w \to \nu(w)|_u$ est un isomorphisme $\mathbb{R}$-linéaire entre $\pi^{-1}(x)$ et $V|_u(\pi)$ avec $\pi(u) = x$.

En théorie de groupes de Lie, puisque l'espace tangent T_xM d'une variété n-dimensionelle lisse connexe M en x est identifié à l'espace Euclidien $\mathbb{R}^n$, alors le groupe $\mathrm{GL}(T_xM)$ de tous les endomorphismes linéaires et inversibles de T_xM est isomorphe au groupe $\mathrm{GL}(n)$ de toutes les matrices réelles et inversibles $n \times n$. Si, de plus, (M, g) est une variété Riemannienne, alors le groupe $\mathrm{O}(n)$ de toutes les transformations g-orthogonales de T_xM en $x \in M$ est isomorphe à $\mathrm{O}(n)$. De même, on a bien $\mathrm{SO}(T_xM) = \mathrm{SO}(n)$ et $\mathfrak{so}(T_xM) = \mathfrak{so}(n)$. On note par $\mathfrak{so}(M) := \bigcup_{x\in M} \mathfrak{so}(T_xM)$ l'ensemble $\{B \in T^*M \otimes TM \mid B^T + B = 0\}$ avec B^T est la matrice transposée de B dans $\mathrm{GL}(n)$.

D'autre part, si $(\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}}$ est l'ensemble de matrices réelles $\hat{n} \times n$, le groupe spécial orthogonal est donné par

$$\mathrm{SO}(n) = \{A \in (\mathbb{R}^n)^* \otimes \mathbb{R}^n \mid \overline{A^T} A = A\overline{A^T} = id_{\mathbb{R}^n}\}.$$

Puisque, dans ma texte, nous nous intéressons à des variétés des dimensions différentes, nous introduisons

$$\mathrm{SO}(n, \hat{n}) := \begin{cases} \{A \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid \overline{A^T} A = id_{\mathbb{R}^n}\}, & if\ n < \hat{n}, \\ \{A \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid A\overline{A^T} = id_{\mathbb{R}^{\hat{n}}}\}, & if\ n > \hat{n}, \\ \mathrm{SO}(n), & if\ n = \hat{n}, \end{cases} \tag{1.2}$$

où A^T est la matrice transposée de A par rapport au produit scalaire de l'espace vectoriel approprié.

Nous donnons la forme matricielle suivante $I_{n,\hat{n}} \in \mathrm{SO}(n,\hat{n})$,

$$I_{n,\hat{n}} = \begin{cases} \begin{pmatrix} \mathrm{id}_{\mathbb{R}^n} \\ 0 \end{pmatrix}, & if\ n \leq \hat{n}, \\ \\ \begin{pmatrix} \mathrm{id}_{\mathbb{R}^{\hat{n}}} & 0 \end{pmatrix}, & if\ n \geq \hat{n}. \end{cases} \tag{1.3}$$

Retournons aux notions de la géométrie différentielle, on note par $T^k_m M$ l'espace des sections de (k,m)-fibré tensoriel de (M,∇). Si $\gamma : I \to M$ est une courbe absolument continue dans M et définie sur un intervalle réel $I \ni 0$, alors le transport parallèle $(P^\nabla)^t_0(\gamma)T_0$ d'un tenseur $T_0 \in T^k_m|_{\gamma(0)}M$ par rapport à ∇ le long de γ en $t \in I$ est la solution unique de l'ODE

$$\nabla_{\dot{\gamma}(t)}((P^\nabla)^t_0(\gamma)T_0) = 0, \quad \text{p.p.}\ \ t \in I,$$

de la condition initiale,

$$(P^\nabla)^0_0(\gamma)T_0 = T_0.$$

Soit $(\hat{M},\hat{\nabla})$ une autre variété affine et $f : M \to \hat{M}$ un lisse morphisme. Alors, on dit que f est affine si pour toute courbe absolument continue $\gamma : [0,1] \to M$, on a

$$f_*|_{\gamma(1)} \circ (P^\nabla)^1_0(\gamma) = (P^{\hat{\nabla}})^1_0(f \circ \gamma) \circ f_*|_{\gamma(0)}. \tag{1.4}$$

On utilise la notation $\mathrm{Aff}(M)$ pour le groupe de toutes les transformations affines et inversibles de la variété M et on l'appelle le groupe affine de M. En particulier, le groupe affine de l'espace $\mathbb{R}^n$ muni de la connexion Euclidienne est noté par $\mathrm{Aff}(n)$. Rappelons aussi que $\mathrm{Aff}(n)$ est égal au produit semi-direct $\mathbb{R}^n \rtimes \mathrm{GL}(n)$ où le produit de groupe $\diamond$ est donné par

$$(v,L) \diamond (u,K) := (Lu + v, L \circ K).$$

En outre, un morphisme lisse $f : M \to \hat{M}$ est une isométrie locale entre deux variété Riemanniennes (M,g) et $(\hat{M},\hat{g})$ si, pour tout $x \in M$, $f_*|_x : T_xM \to T_{f(x)}\hat{M}$ est un morphisme isométrique. En addition, si f est bijectif alors elle est une isométrie et on dit que (M,g) et $(\hat{M},\hat{g})$ sont isométriques. Ces variétés sont localement isométriques s'il existe une variété Riemannienne (N,h) et des isométries locales $F : N \to M$ et $G : N \to \hat{M}$ qui sont aussi des revêtements. Par ailleurs, on utilise $Iso(M,g)$ pour noter le groupe de Lie des isométries de (M,g). De plus, pour toute courbe absolument continue $\gamma : [0,1] \to M$ et $F \in Iso(M,g)$, on a (cf. [32], page 41, Eq. (3.5))

$$F_*|_{\gamma(t)} \circ (P^{\nabla^g})^t_s(\gamma) = (P^{\nabla^g})^t_s(F \circ \gamma) \circ F_*|_{\gamma(s)}, \forall s,t \in [0,1]. \tag{1.5}$$

Dans les équations (2.3) et (2.4), f_* et F_* sont les applications tangentes de f et F respectivement.

D'autre part, on dit d'une courbe $\gamma : [a,b] \to M$ un lacet basé en point $x \in M$ s'il vérifie $\gamma(a) = \gamma(b) = x$. On note par $\Omega_M(x)$ l'ensemble de tous les lacets absolument continus $[0,1] \to M$ basés en point $x \in M$. Pour toutes courbes absolument continues $\gamma : [0,1] \to M$ et $\delta : [0,1] \to M$ sur M satisfaites $\gamma(0) = x$, $\gamma(1) = \delta(0) = y$ et $\delta(1) = z$ où $x, y, z \in M$, on définit l'opération " $\cdot$ " de telle sorte que $\delta \cdot \gamma$ est la courbe absolument continue

$$\delta \cdot \gamma : [0,1] \to M; \quad (\delta \cdot \gamma)(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \delta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} \tag{1.6}$$

Les définitions de transport parallèle et d'ensemble des lacets nous permettent de définir le groupe d'holonomie $H^\nabla|_x$ de M en x par rapport à ∇ comme suivant

$$H^\nabla|_x = \{(P^\nabla)_0^1(\gamma) \mid \gamma \in \Omega_M(x)\}.$$

Pour tout $x \in M$, $H^\nabla|_x$ est un sous-groupe de $GL(T_xM)$, c'est-à-dire le groupe de toutes matrices $n \times n$ inversibles. Dans la situation où M est connexe, on peut joindre deux points quelconques x et y de M par une courbe absolument continue $\gamma : [0,1] \to M$. Ainsi, on aura $(P^\nabla)_0^1(\gamma) H^\nabla|_x (P^\nabla)_1^0(\gamma) = H^\nabla|_y$, ou également, $H^\nabla|_x$ et $H^\nabla|_y$ sont deux sous-groupes conjugués de $GL(T_xM)$. Si (M, g) est une variété Riemannienne et ∇ est la connexion Levi-Civita associée à g alors $H^\nabla|_x$ est un sous-groupe de $\mathrm{O}(T_xM)$. De plus, supposons que M est orientée, alors $H^\nabla|_x$ est un sous-groupe de $\mathrm{SO}(T_xM)$. Prenons un repère orthonormal F de M défini localement en x, on a

$$H^\nabla|_F = \{\mathcal{M}_{F,F}(A) \mid A \in H^\nabla|_x\}.$$

C'est bien un sous-groupe de $\mathrm{SO}(n)$. En particulier, les deux groupes de Lie $H^\nabla|_F$ et $H^\nabla|_x$ sont isomorphes. On notera l'algèbre de Lie du groupe d'holonomie $H^\nabla|_x$ (resp. $H^\nabla|_F$) par $\mathfrak{h}^\nabla|_x$ (resp. $\mathfrak{h}^\nabla|_F$). Ainsi, $\mathfrak{h}^\nabla|_x$ est sous-algèbre de Lie de l'algèbre de Lie $\mathfrak{so}(T_xM)$ et $\mathfrak{h}^\nabla|_F$ est une sous-algèbre de $\mathfrak{so}(n)$.

Fixons deux entiers naturels $k, m \in \mathbb{N}$, l'ensemble de tous les morphismes linéaires $\mathbb{R}^k \to \mathbb{R}^m$ est noté par $\mathcal{L}_k(\mathbb{R}^m)$. On définit $O_k(\mathbb{R}^m)$ l'ensemble de tous $B \in \mathcal{L}_k(\mathbb{R}^m)$ vérifié

(i) Si $k \leq m$, $\| Bu \|_{\mathbb{R}^m} = \| u \|_{\mathbb{R}^k}$ pour tout $u \in \mathbb{R}^k$;

(ii) Si $k \geq m$, B est surjective et $\| Bu \|_{\mathbb{R}^m} = \| u \|_{\mathbb{R}^k}$ pour tout $u \in (kerB)^\perp$ (où $S^\perp$ est le complément orthogonal de $S \subset \mathbb{R}^k$ par rapport à $\langle ., . \rangle_{\mathbb{R}^k}$).

Selon cette définition, on obtient les propositions suivantes

1. $O_k(\mathbb{R}^m)$ est une sous-variété lisse et fermée de $\mathcal{L}_k(\mathbb{R}^m)$.

2. La restriction du morphisme $\mathcal{L}_k(\mathbb{R}^m) \to \mathcal{L}_m(\mathbb{R}^k)$; $A \mapsto A^{\overline{T}}$, où $A^{\overline{T}}$ est la transpose de A, sur $O_k(\mathbb{R}^m) \to O_m(\mathbb{R}^k)$ est un difféomorphisme.

3. Si $k \neq m$, alors $O_k(\mathbb{R}^m)$ est connexe. Si $k = m$, $O_k(\mathbb{R}^k)$ est difféomorphe à $\mathrm{O}(k)$ l'ensemble de $k \times k$-matrices orthogonales.

Ensuite, on va restreindre les définitions et les propriétés précédentes sur une variété Riemannienne (M, g) de dimension n. Donc, pour tout point x de M et tout $k \in \mathbb{N}$, on note par $\mathcal{L}_k(M)|_x$ l'espace des morphismes linéaires $\mathbb{R}^k \to T_xM$. On pose $\mathcal{L}_k(M) := \bigcup_{x\in M} \mathcal{L}_k(M)|_x$.

Definition 1.2.1. *On définit le sous-ensemble $O_k(M)$ de $\mathcal{L}_k(M)$ de tous les éléments $B \in \mathcal{L}_k(M)|_x$, $x \in M$ vérifié*

(i) Si $1 \leq k \leq \dim M$, $\|Bu\|_g = \|u\|_{\mathbb{R}^k}$ pour tout $u \in \mathbb{R}^k$;

(ii) Si $k \geq \dim M$, B est surjective et $\|Bu\|_g = \|u\|_{\mathbb{R}^k}$ pour tout $u \in (\ker B)^\perp$ (où $\perp$ est l'orthogonal par rapport au produit scalaire euclidean dans $\mathbb{R}^k$).

En d'autres termes, $O_k(M)$ est constitué des isométries partielles $\mathbb{R}^k \to T_xM$ qui sont de rang maximal.

Considérons $\pi_{\mathcal{L}_k(M)} : \mathcal{L}_k(M) \to M$; $B \mapsto x$ pour $B \in \mathcal{L}_k(M)|_x$ et $\pi_{O_k(M)} := \pi_{\mathcal{L}_k(M)}|_{O_k(M)} : O_k(M) \to M$.

Proposition 1.2.2. *1. Pour n'importe quel $k \in \mathbb{N}$, $\pi_{\mathcal{L}_k(M)}$ est un fibré vectoriel lisse sur M isomorphe à la somme de Whitney $\bigoplus_{i=1}^k TM \to M$.*

2. Le morphisme $\pi_{O_k(M)}$ est un sous-fibré lisse de $\pi_{\mathcal{L}_k(M)}$ d'une fibre typique égale à $O_k(\mathbb{R}^n)$ où $\mathbb{R}^n$ est munie de la métrique Euclidienne.

3. $O_k(M)$ est connexe pour n'importe quel $k \neq n$.

4. $O_n(M)$ est connexe si M n'est pas orientable.

Pour la preuve des Proposition ci-dessus, voir Part 3, section 9 en [24]).

Dans cette thèse, on désigne par $(\overline{M}, \overline{g}) := (M, g) \times (\hat{M}, \hat{g})$ le produit Riemannien des variétés M et $\hat{M}$ muni par le produit métrique $\overline{g} := g \oplus \hat{g}$. De même, ∇, $\hat{\nabla}$, $\overline{\nabla}$ (resp. R, $\hat{R}$, $\overline{R}$) sont les connections Levi-Civita (resp. les tenseurs de courbure de Riemann) de (M, g), $(\hat{M}, \hat{g})$, $(\overline{M}, \overline{g})$, respectivement. D'après la formule de Koszul, on a,

$$\overline{\nabla}_{(X,\hat{X})}(Y, \hat{Y}) = (\nabla_X Y, \hat{\nabla}_{\hat{X}} \hat{Y}), \quad \forall X,\ Y \in VF(M), \quad \forall \hat{X},\ \hat{Y} \in VF(\hat{M}), \tag{1.7}$$

et,

$$\overline{R}((X, \hat{X}), (Y, \hat{Y}))(Z, \hat{Z}) = (R(X, Y)Z, \hat{R}(\hat{X}, \hat{Y})\hat{Z}), \quad \forall X,\ Y,\ Z \in T_xM, \quad \forall \hat{X},\ \hat{Y},\ \hat{Z} \in T_{\hat{x}}\hat{M}. \tag{1.8}$$

Finalement, on rappelle que si (U, ρ) est une variété Riemannienne munie de la métrique Riemannienne ρ et si (N, ρ_N) est une sous-variété de (U, ρ) munie de la métrique induite de ρ sur N, alors N est totalement géodésique si toute géodésique de (N, ρ_N) est une géodésique de (U, ρ). D'autre part, si (V, ϱ) est une autre variété Riemannienne et $f \in C^\infty(U)$. Définissons la métrique h_f sur $U \times V$ par

$$h_f = pr_1^*(\rho) + (f \circ pr_1)^2 pr_2^*(\varrho),$$

où pr_1 et pr_2 sont les projections dans le premier et le deuxième facteurs du produit $U \times V$ respectievement. Alors, la variété Riemannienne $(U \times V, h_f)$ est dite le produit tordu de (U, ρ) et (V, ϱ) pour la fonction tordue f.

1.3 Les Résultats

1.3.1 Roulement et Développement des variétés

Dans cette section, on va présenter les définitions fondamentales de roulement et de développement des variétés. On commence par fixer deux variétés Riemanniennes connexes complétes orientées (M, g) et $(\hat{M}, \hat{g})$ de dimensions n et $\hat{n}$ respectivement.

Definition 1.3.1. *L'espace d'état $Q = Q(M, \hat{M})$ de roulement de M sur $\hat{M}$ est défini comme suivant:*

(i) si $n \leq \hat{n}$,

$$Q(M, \hat{M}) := \{A \in T^*M \otimes T\hat{M} \mid \hat{g}(AX, AY) = g(X, Y),\ X, Y \in T_xM, x \in M\}.$$

(ii) si $n \geq \hat{n}$,

$$\begin{aligned} Q(M, \hat{M}) := \ &\{A \in T^*M \otimes T\hat{M} \mid \hat{g}(AX, AY) = g(X, Y),\ X, Y \in (\ker A)^{\perp}, \\ &\qquad \mathrm{im}(A) \textit{ est dans l'espace tangent de } \hat{M}\}. \end{aligned}$$

On écrit $q = (x, \hat{x}; A)$ pour un point $q \in Q$. On a $x = \pi_{Q(M,\hat{M}),M}(q)$ et $\hat{x} = \pi_{Q(M,\hat{M}),\hat{M}}(q)$, où

$$\begin{aligned} \pi_{Q(M,\hat{M}),M} &:= \pi_{T^*M\otimes T\hat{M},M}|_{Q(M,\hat{M})} : Q(M, \hat{M}) \to M, \\ \pi_{Q(M,\hat{M}),\hat{M}} &:= \pi_{T^*M\otimes T\hat{M},\hat{M}}|_{Q(M,\hat{M})} : Q(M, \hat{M}) \to \hat{M}. \end{aligned}$$

Ecrivant la matrice transposée de A par rapport à la métrique $(g, \hat{g})$ comme $A^T : T_{\hat{x}}\hat{M} \to T_xM$, on a $(\ker A)^{\perp} = \mathrm{im}(A^T)$. D'où, si $n \leq \hat{n}$ alors $A^TA = \mathrm{id}_{T_xM}$ et si $n \geq \hat{n}$ alors $AA^T = \mathrm{id}_{T_{\hat{x}}\hat{M}}$.

Proposition 1.3.2. *(i) $Q := Q(M, \hat{M})$ est une sous-variété lisse et fermée de $T^*M \otimes T\hat{M}$, sa dimension est égale à*

$$dim(Q) = n + \hat{n} + n\hat{n} - \frac{N(N+1)}{2},\ \textit{where } N := min\{n, \hat{n}\}.$$

De plus, $\pi_{Q,M}$ est un sous-fibré de $\pi_{T^*M\otimes T\hat{M},M}$ de la fibre typique $O_n(\hat{M})$.

(ii) L'application

$$\overline{T}_{M,\hat{M}} : T^*M \otimes T\hat{M} \to T^*\hat{M} \otimes TM;\ \ (x, \hat{x}; A) \mapsto (\hat{x}, x; A^T),$$

est un difféomorphisme et sa restriction sur $Q := Q(M, \hat{M})$ est aussi un difféomorphisme,

$$\overline{T}_Q := \overline{T}_{M,\hat{M}}\ |_Q : Q(M, \hat{M}) \to Q(\hat{M}, M) =: \hat{Q};\ \ \overline{T}_Q(x, \hat{x}; A) = (\hat{x}, x; A^T). \quad (1.9)$$

(*iii*) *Si $n \neq \hat{n}$ ou bien si une de deux variétés M et $\hat{M}$ n'est pas orientable, alors Q est connexe.*

L'application inverse de $\overline{T}_{M,\hat{M}}$ est le morphisme

$$\overline{S} : \hat{Q} \to Q; \quad \overline{S}(\hat{x}, x; B) = (x, \hat{x}; B^T).$$

$\overline{S}$ est bien définie parce que $(\ker B)^{\perp} = \operatorname{im}(B^T)$ et $BB^T = \mathrm{id}_{T_xM}$.

Corollary 1.3.3. *$\pi_{Q(M,\hat{M})} : Q(M, \hat{M}) \to M \times \hat{M}$ est un espace fibré et sa fibre typique est difféomorphe à $O_n(\mathbb{R}^{\hat{n}})$.*

Remark 1.3.4. Si $n \neq \hat{n}$, on peut prouver (*iii*) de Proposition 1.3.2 facilement en utilisant le corollaire précédent avec le fait que $O_n(\mathbb{R}^{\hat{n}})$ est connexe (voir Notation 1.2.2). De toute évidence, on va considérer $\mathrm{SO}(n, \hat{n})$ comme le fibre typique de $\pi_{Q(M,\hat{M})}$.

Proposition 1.3.5. *Soit $q = (x, \hat{x}; A) \in Q$ et soit $B \in T_x^*M \otimes T_{\hat{x}}\hat{M}$. Le champ de vecteurs $\nu(B)|_q$ est tangent à Q, en d'autres mots, il est bien un élément de $V|_q(\pi_Q)$ si et seulement si*

(*i*) *$A^T B \in \mathfrak{so}(T_xM)$, si $n \leq \hat{n}$.*

(*ii*) *$BA^T \in \mathfrak{so}(T_{\hat{x}}\hat{M})$, si $n \geq \hat{n}$.*

Puisque nous nous intéressons en roulement sans pivotement ni glissement, nous écrivons ces conditions géométriquement en prenant d'abord une courbe absolument continue $q : [a, b] \to Q; t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ sur Q, alors

Definition 1.3.6. (*i*) *La courbe $q(\cdot)$ décrive un roulement de M contre $\hat{M}$ sans pivotement si*

$$\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))} A(t) = 0 \ \text{p.p.} \ t \in [a, b]. \tag{1.10}$$

$$\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))} A(t) = 0 \ \text{p.p.} \ t \in [a, b]. \tag{1.11}$$

(*ii*) *La courbe $q(\cdot)$ décrive un roulement de M sur $\hat{M}$ sans glissement si*

$$A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t) \ \text{p.p.} \ t \in [a, b]. \tag{1.12}$$

(*iii*) *La courbe $q(\cdot)$ décrive un roulementde M contre $\hat{M}$ sans glissement ni pivotement si (i) et (ii) sont vérifiés.*

(iii) peut reformuler comme suivant, les courbes $q(\cdot)$ de Q qui décrivent un movement de roulement sans pivotement ni glissement de M contre $\hat{M}$ sont exactement les courbes intégrales du système de commandes affine sans dérive suivant

$$(\Sigma)_R \begin{cases} \dot{\gamma}(t) & = \ u(t), \\ \dot{\hat{\gamma}}(t) & = \ A(t)u(t), \qquad \text{for a.e. } t \in [a, b], \\ \overline{\nabla}_{(u(t),A(t)u(t))} A(t) & = \ 0, \end{cases} \tag{1.13}$$

où le contrôle u est fonction mésurable définie sur un intervalle réel fini $I \subset \mathbb{R}$ et eue des valeurs dans TM.

Afin de définir les distributions de roulement, il est nécessaire de qualifier quelques termes d'où la proposition suivante.

Proposition 1.3.7. *Soit A_0 un (1,1)-tensor sur $M\times\hat{M}$ appartient à $T^1_{1\ (x_0,\hat{x}_0)}(M\times\hat{M})$ avec $(x_0,\hat{x}_0)\in M\times\hat{M}$ et soit $t\mapsto(\gamma(t),\hat{\gamma}(t))$ une courbe absolument continue sur $M\times\hat{M}$ et définie sur l'intervalle réel $I\ni 0$ tel que $\gamma(0)=x_0,\hat{\gamma}(0)=\hat{x}_0$. Alors,*

$$\begin{aligned} A_0\in T^*M\otimes T\hat{M}\Longrightarrow &\quad A(t)=P_0^t(\hat{\gamma})\circ A_0\circ P_t^0(\gamma)\in T^*M\otimes T\hat{M} &&\forall t\in I,\\ A_0\in Q\Longrightarrow &\quad A(t)=P_0^t(\hat{\gamma})\circ A_0\circ P_t^0(\gamma)\in Q &&\forall t\in I.\end{aligned}$$

Les définitions de relèvement et de la distribution de roulement sans pivotement sont déterminées prochainement.

Definition 1.3.8. *(i) Fixons $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$, $X\in T_xM$ et $\hat{X}\in T_{\hat{x}}\hat{M}$, le relèvement de roulement sans pivotement de $(X,\hat{X})$ est l'unique vecteur $\mathscr{L}_{NS}(X,\hat{X})|_q$ de $T_x^*M\otimes T_{\hat{x}}\hat{M}$ en q donné par*

$$\mathscr{L}_{NS}(X,\hat{X})|_q=\frac{d}{dt}\Big|_0 P_0^t(\hat{\gamma})\circ A\circ P_t^0(\gamma)\quad\big(\in T_q(T^*M\otimes T\hat{M})\big),$$

où γ (resp. $\hat{\gamma}$) est une courbe lisse sur M (resp. $\hat{M}$) vérifiant $\gamma(0)=x$ et $\dot{\gamma}(0)=X$ (resp. $\hat{\gamma}(0)=\hat{x}$ et $\dot{\hat{\gamma}}(0)=\hat{X}$).

(ii) La distribution de roulement sans pivotement $\mathscr{D}_{NS}$ sur $T^*M\otimes T\hat{M}$ est une distribution lisse de dimension $(n+\hat{n})$ définie par

$$\mathscr{D}_{NS}|_q=\mathscr{L}_{NS}(T_{(x,\hat{x})}(M\times\hat{M}))|_q.$$

Ainsi, on définit le relèvement et la distribution de roulement sans pivotement et sans glissement.

Definition 1.3.9. *(i) Pour $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$ et $X\in T_xM$, on définit le relèvement de roulement de X comme étant l'unique vecteur $\mathscr{L}_R(X)|_q$ de $T_x^*M\otimes T_{\hat{x}}\hat{M}$ en q suivant*

$$\mathscr{L}_R(X)|_q:=\mathscr{L}_{NS}(X,AX)|_q.$$

(ii) La distribution de roulement $\mathscr{D}_R$ sur $T^*M\otimes T\hat{M}$ est une distribution lisse de dimension égale à n définie en chaque $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$ par

$$\mathscr{D}_R|_q=\mathscr{L}_R(T_x(M)|_q.$$

Par Proposition 1.3.7, $\mathscr{L}_{NS}(X,\hat{X})|_q$ et $\mathscr{L}_R(X)|_q$ appartiennent à T_qQ pour tout $q\in Q$, $X\in T_xM$ et $\hat{X}\in T_{\hat{x}}\hat{M}$ vérifiants les donnés de Définition 1.3.8. Ainsi, $\mathscr{D}_{NS}|_q\subset T_qQ$ et $\mathscr{D}_R|_q\subset T_qQ$ pour tout $q\in Q$. Donc, $\mathscr{D}_{NS}|_Q$ et $\mathscr{D}_R|_Q$ sont distributions lisses sur Q de dimension $n+\hat{n}$ et n respectivement, on les note simplement par $\mathscr{D}_{NS}$ et dr. Certaines propriétés de ces distributions sont collectées dans les corollaires suivantes.

Corollary 1.3.10. *Concernant le roulement sans pivotement, on a les propriétés suivantes,*

(i) *Pour tout* $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ *(resp.* $\in Q$*),* $(\pi_{T^*M\otimes T\hat{M}})_*$ *(resp.* $(\pi_Q)_*$*) est un isomorphisme de* $\mathscr{D}_{NS}|_q$ *à* $T_{(x,\hat{x})}(M \times \hat{M})$*.*

(ii) *Soient* $\overline{X} \in T_{(x,\hat{x})}(M \times \hat{M})$*, A une section locale de* $\pi_{T^*M\otimes T\hat{M}}$ *et* A_* *sa dérivation, alors*

$$\mathscr{L}_{NS}(\overline{X})|_{A(x,\hat{x})} = A_*(\overline{X}) - \nu(\overline{\nabla}_{\overline{X}}A)|_{A(x,\hat{x})}. \tag{1.14}$$

(iii) *Une courbe absolument continue* $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ *dans* $T^*M \otimes T\hat{M}$ *ou Q définie sur un intervalle réel I est tangente à* $\mathscr{D}_{NS}$ *presque partout sur I si et seulement si* $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A = 0$ *presque partout sur I.*

(iv) *On a,* $T(T^*M \otimes T\hat{M}) = \mathscr{D}_{NS} \oplus_{T^*M\otimes T\hat{M}} V(\pi_{T^*M\otimes T\hat{M}})$ *et* $TQ = \mathscr{D}_{NS} \oplus_Q V(\pi_Q)$*.*

Corollary 1.3.11. *Concernant le roulement sans pivotement et sans glissement, on a les propriétés suivantes,*

(i) *Pour tout* $q = (x, \hat{x}; A) \in Q$*,* $(\pi_{Q,M})_*$ *est un isomorphisme de* $\mathscr{D}_R|_q$ *à* T_xM*.*

(ii) *Fixons* $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ *et une courbe absolument continue* $\gamma : [0, a] \to M$*,* $a > 0$ *telle que* $\gamma(0) = x_0$*, alors il existe une unique courbe absolument continue* $q : [0, a'] \to Q$*,* $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$*,* $0 < a' \leq a$*, tangente à* $\mathscr{D}_R$ *presue partout et telle que* $q(0) = q_0$*. On la note par par*

$$t \mapsto q_{\mathscr{D}_R}(\gamma, q_0)(t) = (\gamma(t), \hat{\gamma}_{\mathscr{D}_R}(\gamma, q_0)(t); A_{\mathscr{D}_R}(\gamma, q_0)(t)). \tag{1.15}$$

De plus, si $(\hat{M}, \hat{g})$ *est une variété complète, alors* $a' = a$*.*

(iii) *Si* $q : [0, a] \mapsto Q$*est une courbe tangente à* $\mathscr{D}_R$ *presque partout alors elle peut s'écrire comme* $q_{\mathscr{D}_R}(\gamma, q(0))$ *où* $\gamma = \pi_{Q,M} \circ q$*.*

(iv) *Une courbe absolument continue* $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ *définie sur un intervalle réel I est une courbe de roulement dans Q si et seulement si elle est tangente à* $\mathscr{D}_R$ *presque partout sur I, c'est-à-dire si et seulement si* $\dot{q}(t) = \mathscr{L}_R(\dot{\gamma}(t))|_{q(t)}$ *presue partout sur I.*

Remark 1.3.12. On définit une bijection $\Lambda_x^{\nabla^h}(\cdot)$ entre l'ensemble des courbes a.c. $\gamma : [0, 1] \to M$ commençant du point $x \in M$ et un ouvert de l'espace de Banach des courbes a.c. $[0, 1] \to T_xM$ commençant de 0 comme suivant

$$\Lambda_x^{\nabla}(\gamma)(t) = \int_0^t (P^{\nabla})_s^0(\gamma)\dot{\gamma}(s)ds \ \in T_xM, \ \forall t \in [0, 1].$$

La même définition est appliquée sur $\hat{M}$. L'écriture intrinsèque de la courbe de roulement de conditions initiales (γ, q_0) est:

$$q_{\mathscr{D}_R}(\gamma, q_0)(t) = (\gamma(t), \hat{\Lambda}_{\hat{x}_0}^{-1}(A_0 \circ \Lambda_{x_0}(\gamma))(t); P_0^t(\hat{\Lambda}_{\hat{x}_0}^{-1}(A_0 \circ \Lambda_{x_0}(\gamma))) \circ A_0 \circ P_t^0(\gamma)). \tag{1.16}$$

Si $\widehat{\mathscr{L}_{NS}}$ et $\widehat{\mathscr{L}_R}$ (resp. $\widehat{\mathscr{D}_{NS}}$ et $\widehat{\mathscr{D}_R}$) sont les relèvements de roulement (resp. les distributions de roulement) de $\hat{Q} := Q(\hat{M}, M)$, alors $\dim \widehat{\mathscr{D}_{NS}} = n + \hat{n} = \dim \mathscr{D}_{NS}$ tandis que $\dim \widehat{\mathscr{D}_R} = \hat{n}$ et $\dim \mathscr{D}_R = n$. Par conséquent, le modèle de roulement des variétés M et $\hat{M}$ à dimensions différentes n'est pas symétrique par rapport à l'ordre de M et $\hat{M}$.

Proposition 1.3.13. *Soit $\overline{T} := \overline{T}_Q$ le morphisme défini par* (1.9)*, on a,*

1. $\overline{T}_* \mathscr{D}_{NS} = \widehat{\mathscr{D}_{NS}}$,
2. $\overline{T}_* V(\pi_Q) = V(\pi_{\hat{Q}})$,
3. *lorsque $n \leq \hat{n}$, alors $\overline{T}_* \mathscr{D}_R \subset \widehat{\mathscr{D}_R}$.*

On va présenter les différents types des crochets de Lie sur Q.

Proposition 1.3.14. *Soit $\mathcal{O} \subset T^*M \otimes T\hat{M}$ une sous-variété immergée. Prenons $\overline{T} = (T, \hat{T})$, $\overline{S} = (S, \hat{S}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$ tels que pour tout $q = (x, \hat{x}; A) \in \mathcal{O}$, on a $\mathscr{L}_{NS}(\overline{T}(q))|_q$, $\mathscr{L}_{NS}(\overline{S}(q))|_q \in T_q\mathcal{O}$ et soient U, $V \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ tels que pour tout $q = (x, \hat{x}; A) \in \mathcal{O}$, on a $\nu(U(q))|_q$, $\nu(V(q))|_q \in T_q\mathcal{O}$. Alors,*

1. $$\begin{aligned}[\mathscr{L}_{NS}(\overline{T}(.)), \mathscr{L}_{NS}(\overline{S}(.))]|_q = \ & \mathscr{L}_{NS}(\mathscr{L}_{NS}(\overline{T}(q))|_q\overline{S} - \mathscr{L}_{NS}(\overline{S}(q))|_q\overline{T})|_q \\ & +\nu(AR(T(q), S(q)) - \hat{R}(\hat{T}(q), \hat{S}(q))A)|_q,\end{aligned}$$
2. $$[\mathscr{L}_{NS}(\overline{T}(.)), \nu(U(.))]|_q = -\mathscr{L}_{NS}(\nu(U(q))|_q\overline{T})|_q + \nu(\mathscr{L}_{NS}(\overline{T}(q))|_qU)|_q,$$
3. $$[\nu(U(.)), \nu(V(.))]|_q = \nu(\nu(U(q))|_qV - \nu(V(q))|_qU)|_q.$$

Les deux côtés des Equations 1. , 2. *et* 3. *sont tangents à $\mathcal{O}$.*

On peut généraliser Remarque 1.3.12 pour obtenir un modèle de roulement des variétés affines, c'est qu'on l'appelle le développement de roulement. Soient (M, ∇) et $(\hat{M}, \hat{\nabla})$ deux variétés affines de dimension n avec ∇ et $\hat{\nabla}$ sont connexions affines complètes associées à M et $\hat{M}$ respectivement. On a les définitions suivantes tirées de [10] et [24] concernant le développement des variétés affines.

Definition 1.3.15. *Soit $\gamma : [0, 1] \to M$ une courbe a.c. sur M commençant en x_0. Le développement de γ dans $T_{x_0}M$ par rapport à ∇ est la courbe a.c. $\Lambda^{\nabla}_{x_0}(\gamma) : [0, 1] \to T_{x_0}M$ définie par*

$$\Lambda^{\nabla}_{x_0}(\gamma)(t) = \int_0^t (P^{\nabla})^0_s(\gamma)\dot{\gamma}(s)ds, \qquad t \in [0, 1].$$

Definition 1.3.16. *Soient $(x_0, \hat{x}_0) \in M \times \hat{M}$, $A_0 \in T^*_{x_0}M \otimes T_{\hat{x}_0}\hat{M}$ et $\gamma : [0, 1] \to M$ une courbe a.c. commençant en $\gamma(0) = x_0$. Le développement de γ sur $\hat{M}$ en passant par A_0 par rapport à $\overline{\nabla}$ est la courbe a.c. $\Lambda^{\overline{\nabla}}_{A_0}(\gamma) : [0, 1] \to M$ définie par*

$$\Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t) := (\Lambda^{\hat{\nabla}}_{\hat{x}_0})^{-1}(A_0 \circ \Lambda^{\nabla}_{x_0}(\gamma))(t), \qquad t \in [0, 1].$$

De plus, le transport parallèle relative de A_0 le long de γ par rapport à $\overline{\nabla}$ est le morphisme linéaire suivant

$$(P^{\overline{\nabla}})_0^t(\gamma)A_0 : T_{\gamma(t)}M \to T_{\Lambda_{A_0}^{\overline{\nabla}}(\gamma)(t)}\hat{M}, \text{ such that for } t \in [0,1],$$
$$(P^{\overline{\nabla}})_0^t(\gamma)A_0 := (P^{\hat{\nabla}})_0^t(\Lambda_{A_0}^{\overline{\nabla}}(\gamma)) \circ A_0 \circ (P^{\nabla})_t^0(\gamma).$$

Ainsi,

Definition 1.3.17. *L'espace d'état de développement de roulement de (M, ∇) sur $(\hat{M}, \hat{\nabla})$ est*

$$Q := Q(M, \hat{M}) = \{A \in T_x^*M \otimes T_{\hat{x}}\hat{M} \mid A \in GL(n),\ x \in M \text{ et } \hat{x} \in \hat{M}\}.$$

Un point $q \in Q$ est rédigé comme avant, c'est-à-dire $q = (x, \hat{x}; A)$.

Le relèvement de développement de non-pivotement, la distribution de développement de non-pivotement, le relèvement de développement de roulement et la distribution de développement de roulement sont données comme suivants.

Definition 1.3.18. *Soit $q = (x, \hat{x}; A) \in Q$, $(X, \hat{X}) \in T_{(x,\hat{x})}(M \times \hat{M})$ et γ (resp. $\hat{\gamma}$) une courbe a.c. sur M (resp. sur $\hat{M}$) commençant par x (resp. $\hat{x}$) avec vitesse initiale X (resp. $\hat{X}$). Le relèvement de développement de non-pivotement de $(X, \hat{X})$ est l'unique vecteur $\mathscr{L}_{NS}(X, \hat{X})|_q$ de T_qQ en $q = (x, \hat{x}; A)$ qui vérifie*

$$\mathscr{L}_{NS}(X, \hat{X})|_q := \frac{d}{dt}\Big|_0 (P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma).$$

La distribution de développement de non-pivotement $\mathscr{D}_{NS}$ en $q = (x, \hat{x}; A) \in Q$ est la lisse distribution de dimension $2n$ qui est égale à

$$\mathscr{D}_{NS}|_q := \mathscr{L}_{NS}(T_{(x,\hat{x})}M \times \hat{M})|_q.$$

Si $\hat{X} = AX$, alors le relèvement de développement de roulement en $q \in Q$ est le morphisme injectif $\mathscr{L}_R(X)|_q$

$$\mathscr{L}_R(X)|_q := \mathscr{L}_{NS}(X, AX)|_q = \frac{d}{dt}\Big|_0 (P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma).$$

La distribution de développement de roulement $\mathscr{D}_R$ en $q = (x, \hat{x}; A) \in Q$ est la lisse distribution de dimension n qui est égale à

$$\mathscr{D}_R|_q := \mathscr{L}_R(T_xM)|_q.$$

On dit que la courbe absolument continue $I \ni t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ sur Q est une courbe de développement de roulement si et seulement s'elle est tangente à $\mathscr{D}_R$ presque partout sur I, c'est-à-dire si et seulement si $\dot{q}(t) = \mathscr{L}_R(\dot{\gamma}(t))|_{q(t)}$ presque partout sur I.

Proposition 1.3.19. *Pour tout $q_0 := (x_0, \hat{x}_0; A_0) \in Q$ et toute courbe a.c. $\gamma : [0,1] \to M$ de position initiale x_0, il existe des courbes a.c. et uniques $\hat{\gamma}(t) := \Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t)$ et $A(t) := (P^{\overline{\nabla}})^t_0(\gamma)A_0$ telles que $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$ et $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A(t) = 0$, pour tout $t \in [0,1]$.*

Notons R^{∇} et T^{∇} (resp. $\hat{R}^{\hat{\nabla}}$ et $\hat{T}^{\hat{\nabla}}$) sont le tenseur de courbure et le tenseur de torsion respectivement de M par rapport à ∇ (resp. de $\hat{M}$ par rapport à $\hat{\nabla}$). Alors Proposition 3.7, Lemma 3.18, Proposition 3.24, Proposition 3.26, Proposition 4.1, Proposition 4.6 de [26] nous donnent la proposition suivante.

Proposition 1.3.20. *Soit $\mathcal{O} \subset T^*M \otimes T\hat{M}$ une sous-variété immergée. Prenons $\overline{Z} = (Z, \hat{Z})$, $\overline{S} = (S, \hat{S}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ tels que, pour tout $q = (x, \hat{x}; A) \in \mathcal{O}$, on a $\mathscr{L}_{NS}(\overline{Z}(q))|_q$, $\mathscr{L}_{NS}(\overline{S}(q))|_q \in T_q\mathcal{O}$ et soient U, $V \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ tels que pour tout $q = (x, \hat{x}; A) \in \mathcal{O}$, on a $\nu(U(q))|_q$, $\nu(V(q))|_q \in T_q\mathcal{O}$. Alors,*

$$\mathscr{L}_{NS}(\overline{Z}(A))|_q\overline{S}(\cdot) := \overline{\nabla}_{\overline{Z}(A)}(\overline{S}(A)) - \nu(\overline{\nabla}_{\overline{Z}(A)}A)|_q\overline{S}(\cdot), \tag{1.17}$$

$$\begin{aligned}
[\mathscr{L}_{NS}(\overline{Z}(\cdot)), \mathscr{L}_{NS}(\overline{S}(\cdot))]|_q &= \mathscr{L}_{NS}(\mathscr{L}_{NS}(\overline{Z}(A))|_q\overline{S}(\cdot) - \mathscr{L}_{NS}(\overline{S}(A))|_q\overline{Z}(\cdot))|_q \\
&- \mathscr{L}_{NS}(T^{\nabla}(Z(q), S(q)), \hat{T}^{\hat{\nabla}}(\hat{Z}(q), \hat{S}(q)))|_q \\
&+ \nu(AR^{\nabla}(Z(q), S(q)) - \hat{R}^{\hat{\nabla}}(\hat{Z}(q), \hat{S}(q))A)|_q,
\end{aligned} \tag{1.18}$$

$$\begin{aligned}
[\mathscr{L}_R(Z(\cdot)), \mathscr{L}_R(S(\cdot))]|_q &= \mathscr{L}_R([Z(q), S(q)])|_q \\
&+ \mathscr{L}_{NS}(AT^{\nabla}(Z(q), S(q)) - \hat{T}^{\hat{\nabla}}(AZ(q), AS(q)))|_q \\
&+ \nu(AR^{\nabla}(Z(q), S(q)) - \hat{R}^{\hat{\nabla}}(AZ(q), AS(q))A)|_q,
\end{aligned} \tag{1.19}$$

$$[\mathscr{L}_{NS}(\overline{Z}(\cdot)), \nu(U(\cdot))]|_q = -\mathscr{L}_{NS}(\nu(U(A))|_q\overline{Z}(\cdot))|_q + \nu(\mathscr{L}_{NS}(\overline{Z}(A))|_qU(\cdot))|_q, \tag{1.20}$$

$$[\nu(U(\cdot)), \nu(V(\cdot))]|_q = \nu(\nu(U(A))|_qV - \nu(V(A))|_qU)|_q. \tag{1.21}$$

1.3.2 Des Résultats sur la Commandabilité

Utilisons Equation 1. de Proposition 1.3.14 pour introduire la notion de la courbure de roulement.

Definition 1.3.21. *Pour un point $q = (x, \hat{x}; A) \in Q$, on définit la courbure de roulement Rol_q en q par*

$$\mathsf{Rol}_q(X, Y) := AR(X, Y) - \hat{R}(AX, AY)A, \quad X, Y \in T_xM.$$

Nous écrivons $Rol_q(X, Y) = Rol(X, Y)(A)$.

D'où le crochet

$$[\mathscr{L}_R(X), \mathscr{L}_R(Y)]|_q = \mathscr{L}_R([X,Y])|_q + \nu(\mathsf{Rol}_q(X,Y))|_q.$$

Nous passons maintenant à présenter un principal résultat de la commandabilité du système de roulement sans pivotement de (M,g) sur $(\hat{M},\hat{g})$. Soit $q=(x,\hat{x};A)\in Q$, fixons des repères orthonormaux F et $\hat{F}$ de M et $\hat{M}$ en x et $\hat{x}$ respectivement. On note par $\mathfrak{h} := \mathfrak{h}|_F \subset \mathfrak{so}(n)$ et $\hat{\mathfrak{h}} := \hat{\mathfrak{h}}|_{\hat{F}} \subset \mathfrak{so}(\hat{n})$ les algèbres de Lie des groupes d'holonomie de M et $\hat{M}$ par rapport aux repères F et $\hat{F}$ respectivement.

Theorem 1.3.22. *Soit $q_0=(x_0,\hat{x}_0;A_0)\in Q$, l'intersection de l'orbite $\mathcal{O}_{\mathscr{D}_{NS}}(q_0)$ avec la fibre de π_Q au-dessus de $(x_0,\hat{x}_0)$ vérifie*

$$\mathcal{O}_{\mathscr{D}_{NS}}(q_0)\cap\pi_Q^{-1}(x_0,\hat{x}_0) = \{\hat{h}\circ A_0\circ h \mid \hat{h}\in\hat{H}|_{\hat{x}_0},\ h\in H|_{x_0}\} =: \hat{H}|_{\hat{x}_0}\circ A_0\circ H|_{x_0},$$

Ainsi, sur l'espace tangent, on a

$$\begin{aligned} T_{q_0}\mathcal{O}_{\mathscr{D}_{NS}}(q_0)\cap V|_{q_0}(\pi_Q) &= \nu(\{\hat{k}\circ A_0 - A_0\circ k \mid \quad k\in\mathfrak{h}|_{x_0},\ \hat{k}\in\hat{\mathfrak{h}}|_{\hat{x}_0}\})|_{q_0} \\ &=: \nu(\hat{\mathfrak{h}}|_{\hat{x}_0}\circ A_0 - A_0\circ\mathfrak{h}|_{x_0})|_{q_0}. \end{aligned}$$

Le système de roulement $(\Sigma)_{NS}$ n'est pas commandable dans $T^*M\otimes T\hat{M}$. Tandis que dans Q, on va établir prochainement deux conditions équivalentes à la commandabilité de $(\Sigma)_{NS}$.

Theorem 1.3.23. *Fixons deux bases orthonormales F et $\hat{F}$ de M et $\hat{M}$ en x et $\hat{x}$ respectivement. Alors, le système de roulement sans pivotement $(\Sigma)_{NS}$ de M sur $\hat{M}$ est complètement commandable si et seulement si, pour tout $A\in\mathrm{SO}(n,\hat{n})$, on a*

$$\hat{\mathfrak{h}}A - A\mathfrak{h} = \begin{cases} \{B\in(\mathbb{R}^n)^*\otimes\mathbb{R}^{\hat{n}} \mid A^TB\in\mathfrak{so}(n)\}, & si\ n<\hat{n}, \\ \{B\in(\mathbb{R}^n)^*\otimes\mathbb{R}^{\hat{n}} \mid BA^T\in\mathfrak{so}(\hat{n})\}, & si\ n>\hat{n}. \end{cases}$$

Theorem 1.3.24. *Supposons que (M,g) et $(\hat{M},\hat{g})$ sont simplement connexes. Alors, le système de roulement sans pivotement $(\Sigma)_{NS}$ de M sur $\hat{M}$ est complètement commandable si et seulement si*

$$\hat{\mathfrak{h}}I_{n,\hat{n}} - I_{n,\hat{n}}\mathfrak{h} = \begin{cases} \{B\in(\mathbb{R}^n)^*\otimes\mathbb{R}^{\hat{n}} \mid I_{n,\hat{n}}^TB\in\mathfrak{so}(n)\}, & si\ n\leq\hat{n}, \\ \{B\in(\mathbb{R}^n)^*\otimes\mathbb{R}^{\hat{n}} \mid BI_{n,\hat{n}}^T\in\mathfrak{so}(\hat{n})\}, & si\ n\geq\hat{n}. \end{cases}$$

Concernant la commandabilité du système de roulement sans pivotement ni glissement $(\Sigma)_R$ de (M,g) sur $(\hat{M},\hat{g})$, on a les résultats suivants.

Proposition 1.3.25. *Fixons un $q_0=(x_0,\hat{x}_0;A_0)\in Q$, un X quelconque de $\mathrm{VF}(M)$ et une série réelle $(t_n)_{n=1}^\infty$ qui vérifie $t_n\neq 0$ pour tout n et $\lim_{n\to\infty}t_n=0$. Supposons qu'on a*

$$V|_{\Phi_{\mathscr{L}_R(X)}(t_n,q_0)}(\pi_Q)\subset T(\mathcal{O}_{\mathscr{D}_R}(q_0)),\quad\forall n. \tag{1.22}$$

Alors $\mathscr{L}_{NS}(Y,\hat{Y})|_{q_0}\in T_{q_0}(\mathcal{O}_{\mathscr{D}_R}(q_0))$ pour tout champ de vecteurs Y orthogonal à $X|_{x_0}$ dans $T_{x_0}M$ par rapport à g et tout champ de vecteurs $\hat{Y}$ orthogonal à $A_0X|_{x_0}$ dans $T_{\hat{x}_0}\hat{M}$ par rapport à $\hat{g}$ et appartenu à $A_0(X|_{x_0})^\perp$. Parsuite, la codimension de l'orbite de roulement $\mathcal{O}_{\mathscr{D}_R}(q_0)$ dans Q est plus petite ou égale à $|\hat{n}-n|+1$.

Corollary 1.3.26. *Supposons qu'il existe un point* $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ *et un nombre* $\epsilon > 0$ *tel que pour tout* $X \in \mathrm{VF}(M)$ *satisfaisant* $\|X\|_g < \epsilon$ *sur* M*, on a*

$$V|_{\Phi_{\mathscr{L}_R(X)}(t,q_0)}(\pi_Q) \subset T\mathcal{O}_{\mathscr{D}_R}(q_0), \quad |t| < \epsilon.$$

Alors l'orbite $\mathcal{O}_{\mathscr{D}_R}(q_0)$ *est ouverte dans* Q*. En conséquence,* $(\Sigma)_R$ *est complètement commandable si et seulement si*

$$\forall q \in Q, \quad V|_q(\pi_Q) \subset T_q\mathcal{O}_{\mathscr{D}_R}(q).$$

Soient (M, g) et $(\hat{M}, \hat{g})$ des variétés Riemanniennes de dimensions n et $\hat{n}$ respectivement.

Corollary 1.3.27. *Supposons que* $n \leq \hat{n}$*. Alors, les deux cas suivants sont équivalents,*

(i) $\mathscr{D}_R$ *est involutive,*

(ii) les courbures de (M, g) *et* $(\hat{M}, \hat{g})$ *sont constantes et égales.*

Autrement, si $n > \hat{n}$*, alors les deux cas suivants sont équivalents,*

(a) $\mathscr{D}_R$ *est involutive,*

(b) (M, g) *et* $(\hat{M}, \hat{g})$ *sont des variétés plates.*

Proposition 1.3.28. *Supposons que* (M, g) *et* $(\hat{M}, \hat{g})$ *sont complètes. Les propositions suivantes sont équivalentes*

(i) Il existe un point $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ *tel que* $\mathcal{O}_{\mathscr{D}_R}(q_0)$ *est une variété intégrale de* $\mathscr{D}_R$*.*

(ii) Il existe un point $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ *tel que*

$$Rol_q(X, Y) = 0, \quad \forall q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0), \ X, Y \in T_xM.$$

(iii) Il existe une variété Riemannienne complète (N, h)*, a un revêtement Riemannien* $F : N \to M$ *et un lisse morphisme* $G : N \to \hat{M}$ *vérifiant*

(1) si $n \leq \hat{n}$*,* G *est une immersion Riemannienne dont l'image des* h*-géodésiques par* G *sont des* $\hat{g}$*-géodésiques,*

(2) si $n \geq \hat{n}$*,* G *est une submersion Riemannienne dont la distribution co-noyau* $(\ker G_*)^\perp \subset TN$ *est involutive et les fibres* $G^{-1}(\hat{x})$ *sont variétés totallement géodésiques de* (N, h) *pour tout*$\hat{x} \in \hat{M}$*.*

La proposition et la corollaire suivantes établient des conditions nécessaires de la non-commandabilité du système $\Sigma_{(R)}$ lorsque $n < \hat{n}$.

Proposition 1.3.29. *Soient (M, g) et $(\hat{M}, \hat{g})$ deux variétés Riemanniennes de dimensions n et $\hat{n}$ respectivement avec $n < \hat{n}$. Assumer qu'il existe une sous-variété Riemannienne complète et totallement géodésique $\hat{N}$ de $\hat{M}$ d'une dimension égale à m avec $n \leq m < \hat{n}$. Alors, le système de roulement $\Sigma_{(R)}$ de $Q(M, \hat{M})$ n'est pas complètement controllable.*

Corollary 1.3.30. *Considérons une variété Riemannienne (M, g) de dimension n et une variété Riemannienne $(\hat{M}, \hat{g})$ d'une courbure constante et de dimension $\hat{n}$ avec $n < \hat{n}$. Alors le système de roulement sans pivotement ni glissement de (M, g) sur $(\hat{M}, \hat{g})$ n'est pas controllable.*

1.3.3 Roulement de Variété Riemannienne 2-dimensionnelle sur Variété Riemannienne 3-dimensionnelle

Une application d'un système de roulement des variétés Riemanniennes à deux dimensions différentes est le cas de roulement d'une variété Riemannienne de dimension 2 sur une autre de dimension 3. D'abord, on fixe un $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ puis on définit un repère g-orthonormal local X, $Y \in T_x M$. Notons $\hat{Z}_A := \star(AX \wedge AY) \in T_{\hat{x}} M$, alors AX, AY, $\hat{Z}_A$ forment un repère orthonormal et local dans $T_{\hat{x}} \hat{M}$.

La courbure Gaussienne de M est $K(x) := g(R(X, Y)Y, X)$. Tandis que les courbures sur $\hat{M}$ sont

$$
\begin{aligned}
&\hat{\sigma}_A^1 := \hat{g}(\hat{R}(AY, \hat{Z}_A)\hat{Z}_A, AY) = -\hat{g}(\hat{R}(\star AX), \star AX),\\
&\hat{\sigma}_A^2 := \hat{g}(\hat{R}(AX, \hat{Z}_A)\hat{Z}_A, AX) = -\hat{g}(\hat{R}(\star AY), \star AY),\\
&\hat{\sigma}_A^3 = \hat{\sigma}_A := \hat{g}(\hat{R}(AX, AY)AY, AX) = -\hat{g}(\hat{R}(\star \hat{Z}_A), \star \hat{Z}_A),\\
&\Pi_X(q) := \hat{g}(\hat{R}(\star \hat{Z}_A), \star AX),\\
&\Pi_Y(q) := \hat{g}(\hat{R}(\star \hat{Z}_A), \star AY),\\
&\Pi_Z(q) := \hat{g}(\hat{R}(\star AX), \star AY).
\end{aligned}
$$

Devant ces notations, $Rol_q(X, Y)$ sera égal à

$$
Rol_q(X, Y) = AR(X, Y) - \hat{R}(AX, AY)A = \begin{pmatrix} 0 & -(K - \hat{\sigma}_A) \\ K - \hat{\sigma}_A & 0 \\ \Pi_Y & -\Pi_X \end{pmatrix}.
$$

On a résolu ce problème de commandabilité par distinguer plusieurs cas.

Theorem 1.3.31. *Soient (M, g) et $(\hat{M}, \hat{g})$ deux variétés Riemanniennes de dimensions 2 et 3 respectivement, on rédige les conditions nécessaires de la non-commandabilité du système de roulement sans pivotement ni glissement de (M, g) sur $\hat{M}, \hat{g}$ comme suivant,*

1. *Si $K - \hat{\sigma}_A = 0$ sur un ouvert de $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ et $(\Pi_X, \Pi_Y) = 0$ sur un voisinage de $\hat{x}$, alors $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 2$ et M est isométrique à une sous-variété Riemannienne totalement géodésique de $\hat{M}$.*

2. *Si $K-\hat{\sigma}_A \neq 0$ sur un ouvert de $q=(x,\hat{x};A)\in \mathcal{O}_{\mathscr{D}_R}(q_0)$ et $(\Pi_X,\Pi_Y)=0$ sur un voisinage de $\hat{x}$, alors $\dim \mathcal{O}_{\mathscr{D}_R}(q_0)=5$ et $\hat{M}$ est un produit tordu d'un intervalle réel avec une sous-variété Riemannienne totalement géodésique de dimension 2.*

3. *Si $K-\hat{\sigma}_A = 0$ sur un ouvert de $q=(x,\hat{x};A)\in \mathcal{O}_{\mathscr{D}_R}(q_0)$ et $(\Pi_X,\Pi_Y)\neq 0$ sur un voisinage de $\hat{x}$, alors $\dim \mathcal{O}_{\mathscr{D}_R}(q_0)=7$, M est de courbure constante et $\hat{M}$ ressemble localement au produit Riemannien d'un intervalle réel et une sous-variété Riemannienne de dimension 2.*

4. *Si $K-\hat{\sigma}_A \neq 0$ sur un ouvert de $q=(x,\hat{x};A)\in \mathcal{O}_{\mathscr{D}_R}(q_0)$ et $(\Pi_X,\Pi_Y)\neq 0$ sur un voisinage de $\hat{x}$, alors la dimension de $\mathcal{O}_{\mathscr{D}_R}(q_0)$ est égale à 5 ou 6 ou 7. Si $\dim \mathcal{O}_{\mathscr{D}_R}(q_0)=5$ alors M est une variété plate et et $\hat{M}$ ressemble localement au produit Riemannien d'un intervalle réel et une sous-variété Riemannienne totalement géodésique de dimension 2.*

Concernant le dernier cas du théorème précédent, on n'a pas réussi à trouver les aspect géométriques de M et de $\hat{M}$ lorsque $\dim \mathcal{O}_{\mathscr{D}_R}(q_0)=6$ et lorsque $\dim \mathcal{O}_{\mathscr{D}_R}(q_0)=7$.

1.3.4 Le Groupe d'Holonomie Distributionel

Soit (M,∇) une variété affine et soit Δ une distribution complètement controllable sur M.

Definition 1.3.32. *On définit l'ensemble de lacets absolument continues et Δ-admissible basés en $x\in M$ par*

$$\Omega_\Delta(x) := \{\gamma \mid \gamma:[a,b]\to M \text{ a.c., } \gamma(a)=\gamma(b)=x \text{ et } \dot{\gamma}(t)\in \Delta\big|_{\gamma(t)} \text{ p.p.}\}.$$

Proposition 1.3.33. *Pour chaque $x\in M$, $\Omega_\Delta(x)$ n'est pas vide et il est stable pour la loi " $\cdot$ " donnée dans* (2.5).

On définit le groupe d'holonomie associé à la distribution Δ comme suivant.

Definition 1.3.34. *Le groupe d'holonomie associé à la distribution Δ en un point $x\in M$ est*

$$H^\nabla_\Delta|_x := \{(P^\nabla)^1_0(\gamma) \mid \gamma\in\Omega_\Delta(x)\}.$$

Proposition 1.3.35. *Pour tout points $x\in M$, $H^\nabla_\Delta|_x$ est un sous-groupe de $H^\nabla|_x$. Si y est autre point de M alors $H^\nabla_\Delta|_x$ et $H^\nabla_\Delta|_y$ sont des sous-groupes conjugués.*

La question fondamentale ici, est-ce que la fermeture de H^∇_Δ est égale, en général, à H^∇? La réponse est non et notre preuve est un exemple de roulement de groupe de carnot homogène d'ordre 2 sur un espace Euclidien. Pour celà, on considère le développement de roulement de (M,∇) sur l'espace Euclidien $(\mathbb{R}^n,\hat{\nabla}^n)$ et on désigne la situation à partir des résultats de Section 4 dans [12].

Proposition 1.3.36. *L'espace fibré $\pi_{Q,M} : Q \to M$ est un $\mathrm{Aff}(n)$-fibré principal avec l'action à gauche $\mu : \mathrm{Aff}(n) \times Q \to Q$ donnée par*

$$\mu((\hat{y}, C), q) = (x, C\hat{x} + \hat{y}; C \circ A), \text{ pour } q = (x, \hat{x}; A) \in Q.$$

L'action μ conserve la distribution $\mathscr{D}_R$, c'est-à-dire que pour tout $q \in Q$ et $B \in \mathrm{Aff}(n)$, on a $(\mu_B)_* \mathscr{D}_R|_q = \mathscr{D}_R|_{\mu_B(q)}$ où $\mu_B : Q \to Q$; $q \mapsto \mu(B, q)$. De plus, il existe un sous-groupe unique $\mathcal{H}_q^\nabla$ de $\mathrm{Aff}(n)$ qu'on l'appelle le groupe d'holonomie affine de (M, ∇) et qui vérifie

$$\mu(\mathcal{H}_q^\nabla \times \{q\}) = \mathcal{O}_{\mathscr{D}_R}(q) \cap \pi_{Q,M}^{-1}(x).$$

Si $q' = (x, \hat{x}'; A') \in Q$ appartient à la même fibre de $\pi_{Q,M}$ au-dessus de q, alors $\mathcal{H}_q^\nabla$ et $\mathcal{H}_{q'}^\nabla$ sont conjugués dans $\mathrm{Aff}(n)$ et les classes de conjugaison de $\mathcal{H}_q^\nabla$ sont de la forme $\mathcal{H}_{q'}^\nabla$. Cette classe de conjugaison est notée par $\mathcal{H}^\nabla$.

D'après [27] et le produit semi-direct $\mathrm{Aff}(n) = \mathbb{R}^n \rtimes \mathrm{GL}(n)$, on a le prochain corollaire.

Corollary 1.3.37. *La projection du groupe d'holonomie affine $\mathcal{H}_q^\nabla$ de (M, ∇) sur* $\mathrm{SO}(n)$ *est le groupe d'holonomieH^∇ de (M, ∇).*

Considérons maintenant une distribution lisse complètement commandable Δ sur (M, ∇).

Definition 1.3.38. *La distribution de roulement Δ_R de Δ est la sous-distribution lisse de $\mathscr{D}_R$ définie en $(x, \hat{x}; A) \in Q$ par*

$$\Delta_R|_{(x,\hat{x};A)} = \mathscr{L}_R(\Delta|_x)|_{(x,\hat{x};A)}. \tag{1.23}$$

Puisque Δ est complètement commandable, on a,

Corollary 1.3.39. *Pour tout $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ et toute courbe $\gamma : [0, 1] \to M$ a.c. et Δ-admissible qui commence en x_0, il existe une unique courbe $q_{\Delta_R}(\gamma, q_0) : [0, 1] \to Q$ a.c. et Δ_R-admissible.*

Corollary 1.3.40. *L'action μ mentionée dans Proposition 6.1.7 conserve la distribution Δ_R. De plus, il existe un sous-groupe unique $\mathcal{H}_{\Delta_R|q}^\nabla$ de $\mathcal{H}_q^\nabla$ qu'on l'appelle le groupe d'holonomie affine sur Δ_R et qui vérifie*

$$\mu(\mathcal{H}_{\Delta_R|q}^\nabla \times \{q\}) = \mathcal{O}_{\Delta_R}(q) \cap \pi_{Q,M}^{-1}(x),$$

où $\mathcal{O}_{\Delta_R}(q_0)$ est l'orbite associée à la distribution Δ_R en q_0.

Comme ce qui précède, on a le corollaire suivant d'après [27].

Corollary 1.3.41. *La projection du groupe d'holonomie horizontal affine $\mathcal{H}_{\Delta_R}^\nabla$ sur Δ_R sur* $\mathrm{SO}(n)$ *est le groupe d'holonomie horizontal H_Δ^∇ sur Δ.*

Proposition 1.3.42. *Pour tout $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, la restriction de $\pi_{Q,M} : Q \to M$ sur l'orbite $\mathcal{O}_{\Delta_R}(q_0)$ est une submersion dans M.*

Corollary 1.3.43. *Pour tout $x \in M$, le fibre $\pi_{Q,M}^{-1}(x) \cap \mathcal{O}_{\Delta_R}(q_0)$ de $\mathcal{O}_{\Delta_R}(q_0)$ au-dessus de x est vide ou bien une sous-variété plongée et fermée dans $\mathcal{O}_{\Delta_R}(q_0)$ de dimension $\delta = \dim \mathcal{O}_{\Delta_R}(q_0) - \dim M$.*

Le résultat principal de la dernière partie de la thèse est le suivant.

Proposition 1.3.44. *Le groupe d'holonomy horizontal H_{Δ}^{∇} est un sous-groupe de Lie de $\mathrm{GL}(n)$ et le groupe d'holonomie horizontal affine $\mathcal{H}_{\Delta_R}^{\nabla}$ est un sous-groupe de Lie de $\mathrm{Aff}(n)$.*

On donne la définition de groupe de Carnot homogène d'ordre 2 sur $\mathbb{R}^N$ comme elle est citée dans [5].

Definition 1.3.45. *A. Soit $\circ$ un loi de groupe sur $\mathbb{R}^N$, le groupe $(\mathbb{R}^N, \circ)$ est un groupe de Lie sur $\mathbb{R}^N$ si l'application*

$$\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N; \quad (x, y) \mapsto y^{-1} \circ x$$

est un lisse morphisme.

B. On dit que $(\mathbb{R}^N, \circ)$ est un groupe de Lie homogène s'il existe N-uplet des nombres réels $\sigma = (\sigma_1, ..., \sigma_N)$ vérifiant $1 \leq \sigma_1 \leq ... \leq \sigma_N$ et tels que la dilatation

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N; \quad \delta_\lambda(x_1, ..., x_N) = (\lambda^{\sigma_1} x_1, ..., \lambda^{\sigma_N} x_N)$$

est un automorphisme de groupe pour tout réel $\lambda > 0$. On note ce groupe de Lie homogène par $(\mathbb{R}^N, \circ, \delta_\lambda)$.

C. On dit que le groupe de Lie $\mathbb{G} := (\mathbb{R}^N, \circ)$ est groupe de Carnot homogène si les deux conditions suivantes sont satisfaites,

C-1. $\mathbb{R}^N$ peut être écrire comme $\mathbb{R}^N = \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_r}$ et la dilatation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$,

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, ..., x_{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)} ..., \lambda^r x^{(r)}), \ \textit{pour } x^{(i)} \in \mathbb{R}^{N_i},$$

est un automorphisme de groupe pour tout réel $\lambda > 0$.

C-2. Notons l'algèbre de Lie de $\mathbb{G}$ comme $\mathfrak{g}$. Prenons les champs de vecteurs Z_i, pour $i = 1, ..., N_1$, convenus avec les champs de vecteurs canoniques sur $\mathbb{R}^N$, c'est-à-dire les $\frac{\partial}{\partial x_i}$. On a

$$Lie(Z_1, ..., Z_{N_1}) = \mathfrak{g},$$

où $Lie(Z_1, ..., Z_{N_1})$ est l'algèbre de Lie engendré par les vecteurs Z_i, $i = 1, ..., N_1$.

On constate d'après le roulement du groupe de Carnot $\mathbb{G}$ homogène d'ordre 2 et de dimension N sur l'espace Euclidien $\mathbb{R}^N$ le suivant.

Proposition 1.3.46. *Si $\mathbb{G}$ est un groupe de Carnot homogène d'ordre 2, de dimension $N \geq 3$ et muni de la connexion de Levi-Civita ∇^g associée à une métrique Riemannienne g, alors,*

a) *Le groupe d'holonomie horizontal affine $\mathcal{H}_{\Delta_R}^{\nabla}$ est un sous-groupe de $\mathrm{SE}(N)$ de dimention $2N$,*

b) *Le groupe d'holonomie horizontal H_{Δ}^{∇} est un sous-groupe de Lie compact et connexe de $\mathrm{SO}(N)$ de dimention N,*

c) *Les inclusions $\mathcal{H}_{\Delta_R}^{\nabla} \subset \mathrm{SE}(m+n)$ et $H_{\Delta}^{\nabla} \subset \mathrm{SO}(m+n)$ sont strictes si et seulement si $N > 3$.*

Chapter 2

Introduction and Notations

2.1 Motivations

In this thesis, we are interested in the rolling of two Riemannian connected manifolds (M, g) and $(\hat{M}, \hat{g})$ of dimensions n and $\hat{n}$ respectively. Two constraints are considered on the rolling system: the no-spinning and the no-slipping conditions. The majority of the studies treat the case where the manifolds have equal dimensions (cf. [8, 9, 11]).

To gain further insight about this model, let us consider the rolling problem of two convex surfaces on one another as well as the rolling of a ball (the unit sphere S^2 for example) on a plane in the Euclidean space $\mathbb{R}^3$. When the both surfaces are in contact at some point, then their exterior normal vectors are opposite one another. Such rolling without slipping requires that, at the contact point, the tangent velocity vector to the first surface is equal to the tangent velocity vector to the second surface rotated through an angle $\theta(\cdot)$. The no-spinning condition means that the rotational axes of the bodies are remaining in the common tangent plane which implies a condition on $\dot{\theta}(\cdot)$. Thus, the state space of rolling without twisting nor slipping of two surfaces has dimension equal to five: two points fixed on each surface and an angle θ (cf. [1]).

In general, the state space Q of rolling of two differential manifolds M and $\hat{M}$ is a smooth bundle over the tensorial product space of the cotangent space of M with the tangent space of $\hat{M}$. In other words, its typical fiber is the space of the isometries A between the tangent spaces of the considered manifolds. Geometrically, the rolling without spinning means that the image of a parallel vector fields along a curve in M by A is a parallel vector field along a curve in $\hat{M}$. Moreover, the rolling distribution without spinning $\mathscr{D}_{NS}$ on Q is defined to be the space of the derivation of parallel transport of A along the curve $(x(\cdot), \hat{x}(\cdot))$ in $M \times \hat{M}$. The rolling distribution $\mathscr{D}_R$ which describes the two constraints is a subdistribution of $\mathscr{D}_{NS}$ obtained by taking $A(\cdot)\dot{x}(\cdot) = \dot{\hat{x}}(\cdot)$. Thus, we define two driftless affine control dynamic systems $(\Sigma)_{NS}$ and $(\Sigma)_R$ on Q associated with $\mathscr{D}_{NS}$ and $\mathscr{D}_R$ respectively.

The issue we address here is the controllability for both system $(\Sigma)_{NS}$ and $(\Sigma)_R$ by using geometric tools on M and $\hat{M}$. Precisely, we aim at getting necessary and/or the sufficient conditions so that any pair of two points (q_{init}, q_{final}) in Q, there exists a curve $q(\cdot)$ tangent to $\mathscr{D}_{NS}$ (to $\mathscr{D}_R$ respectively) which steers respectively the system $(\Sigma)_{NS}$ ($(\Sigma)_R$ respectively) from (q_{init} to q_{final}). This allows us to investigate the geometrical features of the set of the end points reached by all the curves $q(\cdot)$ starting at some initial point. It is called the rolling orbit of $\mathscr{D}_{NS}$ (of $\mathscr{D}_R$ respectively). We say that the rolling system is completely controllable if all the orbits are equal to the state space.

In the statement of the above example, if M and $\hat{M}$ are 2-dimensional Riemannian manifolds, the rolling system $(\Sigma)_R$ is completely controllable if and only if the mentioned manifolds are not isometric. More precisely, its rolling orbit is of dimension either 2 or 5. These numbers are according to the difference between the values of the Riemannian curvatures of M and $\hat{M}$ at the based point in Q (cf. [1]). Concerning the rolling of two 3-dimensional Riemannian manifolds, [11] gave all the possibilities of necessary conditions for the non-controllability of $(\Sigma)_R$: the manifolds are either locally isometric, locally of class $\mathcal{M}_\beta$ for some $\beta > 0$ or locally isometric to warped products of real intervals with 2-dimensional Riemannian manifolds (see Section 5.4 for the definitions of class $\mathcal{M}_\beta$ manifolds and warped product of manifolds). Then [12] considered the rolling problem without spinning nor slipping of a smooth connected oriented complete Riemannian manifold (M, g) onto a space form $(\hat{M}, \hat{g})$ of same dimension $n \geq 2$. In particular, if $\hat{M}$ has zero curvature, then $(\Sigma)_R$ is completely controllable if and only if the holonomy group of M with respect to the Levi-Civita connection is equal to the special orthogonal group $\mathrm{SO}(n)$. The second author explained in [24] and [26] the development of affine manifolds which are not necessarily equipped with torsion-free connections.

In the current thesis, Chapter 2 presents in two sections the definitions of the state space, the lifts and the distributions of rolling system of two smooth connected Riemannian manifolds and of rolling development of affine manifolds. In the first part, we generalize [10] to the case where (M, g) and $(\hat{M}, \hat{g})$ have different dimensions n and $\hat{n}$ respectively. For example, in order to roll safely M on $\hat{M}$ when $n \geq \hat{n}$, we must define the state space Q by the isometries A whose are onto the tangent space of $\hat{M}$ at a fixed point $\hat{x}$. In the second part, using the development of affine manifolds which defined in [24] and the rolling lift of $\mathscr{D}_{NS}$ with respect to an arbitrary linear connection as in [26], we define the rolling development of affine manifolds.

In Chapter 3, we introduce the main controllability results on rolling Riemannian manifolds of different dimensions (M, g) and $(\hat{M}, \hat{g})$. Concerning the system $(\Sigma)_{NS}$, due to the studies in [8, 9], the geometrical aspect of the rolling orbits of $\mathscr{D}_{NS}$ is clearly related to the characterization of holonomy groups of M and $\hat{M}$. In the case of $(\Sigma)_R$, the situation is more complicated. Indeed, the rolling distribution $\mathscr{D}_R$ has dimension equal to n, which implies that the rolling problem without spinning nor slipping is not symmetric with respect to the order of the manifolds. Thus, the standard strategy consists in computing the iterated Lie brackets of sections of $\mathscr{D}_R$ and in verifying whether they span the tangent space of Q at each point. The rolling curvature tensor Rol given by the first Lie bracket on $\mathscr{D}_R$, which can be seen as the difference between the Riemannian curvature tensors of M and $\hat{M}$. This means that the controllability of $(\Sigma)_R$ is directly related to the values of the above curvature tensors and their covariant derivatives. In this statement, there is an interesting result that if $n < \hat{n}$ strictly and M has constant Riemannian curvature then the rolling problem without spinning nor slipping is not controllable. When $\mid n - \hat{n} \mid = 1$, one can look to the problem as the rolling of two Riemannian manifolds of same dimensions by taking the Riemannian product with $(\mathbb{R}, s)$, where s is the Euclidean metric. Thus, we present the necessary conditions for the non-controllability in the case $(n, \hat{n}) = (3, 2)$.

On the other side, when $(n, \hat{n}) = (2, 3)$, we are not able to find all necessary conditions for the non-controllability. We prove in Chapter 4 that either $\mathscr{D}_R$ is involutive and M is isometric to a totally geodesic submanifold of $\hat{M}$ or $\hat{M}$ is the warped product of a real interval with 2-dimensional totally geodesic submanifold or M has constant curvature and $\hat{M}$ is locally isometric to the Riemannian product of a real interval with 2-dimensional submanifold. The computations also shows that the dimension of the orbits belongs to the set $\{2, 5, 6, 7\}$.

The last chapter consists of the definition of horizontal holonomy group H^{∇}_{Δ} of a completely controllable distribution Δ on an affine manifold M with respect to an affine connection ∇. Moreover, by using the development of affine manifolds, we give the definitions of the affine holonomy group $\mathcal{H}^{\nabla}$ on (M, ∇) and the definition of the affine horizontal holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ on Δ. They are subgroups of the group $\mathrm{Aff}M$ of all invertible affine transformations from the M onto itself. We then prove that H^{∇}_{Δ} is a Lie subgroup of $\mathrm{GL}(n)$ and $\mathcal{H}^{\nabla}_{\Delta_R}$ is a Lie subgroup of $\mathrm{Aff}(n)$. Determining necessary and (or) sufficient conditions on a completely controllable distribution Δ of M so that the H^{∇}_{Δ} equals H^{∇} is not an obvious question, besides trivial cases. Our second main result is an explicit example for a strict inequality in $\dim(H^{\nabla}_{\Delta}) \leq \dim(H^{\nabla})$. More precisely, M is a free step-two homogeneous Carnot group, ∇^g is the Levi-Civita connection associated with a Riemannian metric g on M. The result is obtained from that fact that the affine holonomy groups have the same geometric structure as the orbits of the rolling development of (M, ∇^g) against $(\mathbb{R}^n, \hat{\nabla}^n)$, where $\hat{\nabla}^n$ is the Euclidean connection.

2.2 Notations

Along this thesis, we denote the element of the i-th row and j-th column for a real matrix A by A^i_j and its usual transpose by A^T. Furthermore, let $L : V \to W$ be a $\mathbb{R}$-linear map where V and W are two $\mathbb{R}$-linear spaces with dimensions n and n' respectively. Taking $F = (v_i)_{i=1}^n$ and $G = (w_i)_{i=1}^{n'}$ two bases of V and W respectively, the $(n' \times n)$- real matrix of L w.r.t. F and G is denoted by $\mathcal{M}_{F,G}(L)$ and given by $L(v_i) = \sum_j \mathcal{M}_{F,G}(L)^j_i w_j$. Furthermore, $T^*_x M \otimes T_{\hat{x}}\hat{M}$ is canonically identified with the linear space of the $\mathbb{R}$-linear map $A : T_x M \to T_{\hat{x}}\hat{M}$. If moreover g and h are inner products of V and W respectively, then $L^{T_{g,h}} : W \to V$ is the transpose of L with respect to g and h, i.e. $g(L^{T_{g,h}}w, v) = h(w, Lv)$. Thus, we can write $(\mathcal{M}_{F,G}(L))^T = \mathcal{M}_{F,G}(L^{T_{g,h}})$.

In the sequel, all manifolds considered are finite dimensional, smooth and connected. Recall that if E, F are smooth manifolds, a smooth bundle $\pi_{E,M} : E \to M$ is a smooth map such that for every $x \in M$ there exists a neighbourhood U of x in M and a smooth diffeomorphism $\tau : \pi_{E,M}^{-1}(U) \to U \times F$ so that $pr_1 \circ \tau = (\pi_{E,M}|_{\pi(U)})^{-1}$, where pr_1 stands for the projection onto the first factor. Then, F is called the typical fiber of $\pi_{E,M}$ and τ is a (smooth) local trivialization of $\pi_{E,M}$. Moreover, the typical fiber F is unique up to diffeomorphism and in the case where F is a finite dimensional $\mathbb{R}$-linear space, we get a (smooth) vector bundle. Moreover, the set $E|_x = \pi_{E,M}^{-1}(x) := \pi_{E,M}^{-1}(x)$ is called the $\pi_{E,M}$-fiber over x. A smooth section of a bundle $\pi_{E,M}$ is a smooth map $s : M \to E$ that satisfies $\pi_{E,M} \circ s = \mathrm{id}_M$ where the set of smooth sections of π is usually denoted by $\Gamma(E)$. When the context is clear, we simply write π for $\pi_{E,M}$. Besides, let G be a Lie group, then a smooth bundle $\pi : E \to M$ is a principal G-bundle over M if there exists a smooth and free action of G on E which preserves the fibers of π, cf. [23].

Let $\Gamma(E)$ be the space of smooth sections of a smooth vector bundle E, then a linear connection on E is an $\mathbb{R}$-linear map $\Gamma(E) \to \Gamma(E \times T^\star M)$ satisfying the Leibniz rule for all smooth functions in $C^\infty(M)$. Let $\mathcal{X}(M)$ be the set of smooth sections of the tangent bundle of M. Thus, an affine connection ∇ on M is a $\mathbb{R}$-bilinear map

$$\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M); \quad (X, Y) \mapsto \nabla_X Y,$$

such that it is $C^\infty(M)$-linear in the first variable and verifies the Leibniz rule over $C^\infty(M)$ in the second variable and (M, ∇) is said to be an affine manifold. If, moreover, the exponential map exp_x^∇ of (M, ∇) is defined on the whole tangent space $T_x M$ for all $x \in M$, then (M, ∇) is said to be geodesically complete affine manifold. We use ∇^n and ∇^g respectively to denote the Euclidean connection on $\mathbb{R}^n$ and the Levi-Civita connection of a Riemannian manifold (M, g) when M is endowed a Riemannian metric g. In this case, (M, g) is assumed to be complete and oriented and we use $\|v\|_g$ to denote $g(v, v)^{1/2}$ for every $v \in T_x M$ at $x \in M$.

We keep using the notation $[\cdot, \cdot]$ for the Lie bracket operation in TM such we define the curvature tensor R^∇ and the torsion tensor of an affine connection ∇ as

$$R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \; T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

respectively, for smooth vector fields X, Y, Z on M.

A distribution Δ over a manifold M is a smooth assignment $x \mapsto \Delta|_x$ where $\Delta|_x \subset T_x M$. The flag of Δ is a collection of distributions Δ^s, $s \geq 1$, satisfying $\Delta^1 \subset \Delta^2|_x \subset \cdots \subset \Delta^r|_x \subset \ldots$ for every $x \in M$ and such that $\Delta^1|_x := \Delta|_x$ and $\Delta^{s+1}|_x := \Delta^s|_x + [\Delta^1, \Delta^s]|_x$ for $s \geq 1$. We say that the distribution Δ on M is of constant rank $m \leq n$ if $\dim(\Delta|_x) = m$ for every $x \in M$ and completely controllable if, for any $x \in M$, there exists an integer $r = r(x)$ such that $\Delta^r|_x = T_x M$. The number $r(x)$ is called the step of $\Delta|_x$ (cf. [19] for more details). An absolutely continuous curve $\gamma : I \to M$ defined on an bounded interval $I \subset \mathbb{R}$ is said to be Δ-admissible curve if it is tangent to Δ almost everywhere (a.e.), i.e., if for a.e. $t \in I$, $\dot\gamma(t) \in \Delta|_{\gamma(t)}$. For $x_0 \in M$, the Δ-orbit through x_0, denoted $\mathcal{O}_\Delta(x_0)$, is the set of endpoints of the Δ-admissible curves of M starting at x_0, i.e.,

$$\mathcal{O}_\Delta(x_0) = \{\gamma(1) \mid \gamma : [0, 1] \to M, \text{ a.c. } \Delta\text{-admissible curve}, \; \gamma(0) = x_0\}.$$

By the Orbit Theorem (cf.[20]), it follows that $\mathcal{O}_\Delta(x_0)$ is an immersed smooth submanifold of M containing x_0 so that the tangent space $T_x\mathcal{O}_\Delta(x_0)$ for every $x \in \mathcal{O}_\Delta(x_0)$ contains $Lie_x(\Delta)$, the evaluation at $x \in M$ of the Lie algebra generated by Δ. Furthermore, if a smooth distribution Δ' on M is a subdistribution of Δ (i.e., $\Delta' \subset \Delta$), then $\mathcal{O}_{\Delta'}(x_0) \subset \mathcal{O}_\Delta(x_0)$ for all $x_0 \in M$. If Δ is completely controllable, then, for every $x \in M$, we have $\mathcal{O}_\Delta(x) = M$ i.e. any two points of M can be joined by an a.c. Δ-admissible curve. Recall that, the Lie Algebra Rank Condition (LARC), i.e. $Lie_x(\Delta) = T_xM$, is a sufficient condition for the complete controllability of Δ (cf. [19]).

We define the vertical distribution $V(\pi)$ on the smooth bundle $\pi : E \to M$ at every $y \in E$ by $V|_y(\pi)$ is the set of all $Y \in T|_yE$ such that $\pi_*(Y) = 0$. Moreover, $\pi_{V(\pi)} := \pi_{TE}|_{V(\pi)}$ defines a vector subbundle of $\pi_{TE} : TE \to E$. If $\pi : E \to M$ is a vector bundle and $\eta : F \to M$ is another vector bundle, we denote by $C^\infty(\pi, \eta)$ the set of smooth maps $g : E \to F$ such that $\eta \circ g = \pi$. Given $f \in C^\infty(\pi, \eta)$ and $u, w \in \pi^{-1}(x)$, the vertical derivative of f at u in the direction w is defined as

$$\nu(w)|_u(f) := \frac{d}{dt}|_0 f(u + tw) \in \nu^{-1}(x).$$

According to this definition, since $\nu(w)|_u$ is an element of $V|_u(\pi)$, then $w \to \nu(w)|_u$ is an $\mathbb{R}$-linear isomorphism from $\pi^{-1}(x)$ onto $V|_u(\pi)$ with $\pi(u) = x$.

In the theory of Lie groups, since the tangent space T_xM is identified to the Euclidean space $\mathbb{R}^n$, then the group $\mathrm{GL}(T_xM)$ of bounded linear invertible endomorphisms of T_xM is clearly isomorphic to the group $\mathrm{GL}(n)$ of $n \times n$ invertible matrices with real entries. If, moreover, (M, g) is a Riemannian manifold, then the group $\mathrm{O}(n)$ of all g-orthogonal transformations of T_xM at $x \in M$ is isomorphic to $\mathrm{O}(n)$. Similarly, this allows one to write $\mathrm{SO}(T_xM) = \mathrm{SO}(n)$ and $\mathfrak{so}(T_xM) = \mathfrak{so}(n)$. We also denote $\mathfrak{so}(M) := \bigcup_{x\in M} \mathfrak{so}(T_xM)$ as the set $\{B \in T^*M \otimes TM \mid B^T + B = 0\}$ where B^T is the usual transpose of B in $\mathrm{GL}(n)$.

Moreover, for any two positive integers n and $\hat{n}$, we denote by $(\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}}$ the set of $\hat{n} \times n$ real matrices and we define

$$\mathrm{SO}(n, \hat{n}) := \begin{cases} \{A \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid A^TA = \mathrm{id}_{\mathbb{R}^n}\}, & if\ n < \hat{n}, \\ \{A \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid AA^T = \mathrm{id}_{\mathbb{R}^{\hat{n}}}\}, & if\ n > \hat{n}, \\ \mathrm{SO}(n), & if\ n = \hat{n}. \end{cases} \tag{2.1}$$

We define the matrix $I_{n,\hat{n}} \in \mathrm{SO}(n, \hat{n})$ as follows,

$$I_{n,\hat{n}} = \begin{cases} \begin{pmatrix} \mathrm{id}_{\mathbb{R}^n} \\ 0 \end{pmatrix}, & if\ n \le \hat{n}, \\ \\ \begin{pmatrix} \mathrm{id}_{\mathbb{R}^{\hat{n}}} & 0 \end{pmatrix}, & if\ n \ge \hat{n}. \end{cases} \tag{2.2}$$

Return to the notions in differential geometry, we state T^k_mM to be the set of sections of the (k, m)-tensor bundle on the affine manifold (M, ∇). If $\gamma : I \to M$ is any a.c. (a.c. for short) curve in M defined on a real interval I containing 0, we use $(P^\nabla)^t_0(\gamma)T_0$,

$t \in I$, to denote the ∇-parallel transport along γ of a tensor $T_0 \in T^k_m|_{\gamma(0)}M$. It is the unique solution for the Cauchy problem

$$\nabla_{\dot\gamma(t)}((P^{\nabla})^t_0(\gamma)T_0) = 0, \quad \text{for a.e. } \ t \in I, \ (P^{\nabla})^0_0(\gamma)T_0 = T_0.$$

Let $(\hat M, \hat\nabla)$ be another affine manifold and $f : M \to \hat M$ be a smooth map. we say that f is affine if for any a.c. curve $\gamma : [0,1] \to M$, one has

$$f_*|_{\gamma(1)} \circ (P^{\nabla})^1_0(\gamma) = (P^{\hat\nabla})^1_0(f \circ \gamma) \circ f_*|_{\gamma(0)}. \tag{2.3}$$

We use $\mathrm{Aff}(M)$ to denote the affine group of all invertible affine transformations from the affine manifold M onto itself. In particular, the affine group of $\mathbb{R}^n$ is denoted by $\mathrm{Aff}(n)$. We recall that the affine group $\mathrm{Aff}(n)$ is equal to $\mathbb{R}^n \rtimes \mathrm{GL}(n)$ and its product group $\diamond$ is given by

$$(v, L) \diamond (u, K) := (Lu + v, L \circ K).$$

Besides, if (M, g) and $(\hat M, \hat g)$ are Riemannian manifolds equipped with Riemannian metrics g and $\hat g$ respectively. Then, the smooth map $f : M \to \hat M$ is a local isometry between M and $\hat M$ if, for all $x \in M$, $f_*|_x : T_xM \to T_{f(x)}\hat M$ is an isometric linear map. If moreover f is bijective, it is called an isometry, and (M, g), $(\hat M, \hat g)$ are said to be isometric. However, (M, g) and $(\hat M, \hat g)$ are locally isometric if there exist a Riemannian manifold (N, h) and local isometries $F : N \to M$ and $G : N \to \hat M$ which are also covering maps. We use $Iso(M, g)$ to denote the smooth Lie group of isometries of (M, g), i.e. it is the set of all diffeomorphism $F : M \to M$ such that $F_*|_x : T_xM \to T_xM$ is an isometry for all $x \in M$. Now, if $\gamma : [0,1] \to M$ is an a.c. curve, then, for any $s, t \in [0,1]$ and $F \in Iso(M, g)$, we have,

$$F_*|_{\gamma(t)} \circ (P^{\nabla^g})^t_s(\gamma) = (P^{\nabla^g})^t_s(F \circ \gamma) \circ F_*|_{\gamma(s)}, \tag{2.4}$$

(cf. [32], page 41, Eq. (3.5)).

We mean here by f_* and F_* in (2.3) and (2.4) the push-forward associated with f and F respectively.

An a.c. curve $\gamma : [a, b] \to M$ is a loop based at $x \in M$ if $\gamma(a) = \gamma(b) = x$. We denote by $\Omega_M(x)$ the space of all a.c. loops $[0,1] \to M$ based at some given point $x \in M$. Moreover, if $\gamma : [0,1] \to M$ and $\delta : [0,1] \to M$ are two a.c. curves on M such that $\gamma(0) = x$, $\gamma(1) = \delta(0) = y$ and $\delta(1) = z$ where $x,\ y,\ z \in M$, the concatenation $\delta \cdot \gamma$ is the a.c. curve defined by

$$\delta \cdot \gamma : [0,1] \to M, \quad (\delta \cdot \gamma)(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}], \\ \delta(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases} \tag{2.5}$$

The previous definitions of parallel transport and $\Omega_M(x)$ allow us to state the next definition of the holonomy group at $x \in M$.

Definition 2.2.1. *For every $x \in M$, the holonomy group $H^{\nabla}|_x$ at x is defined by*

$$H^{\nabla}|_x = \{(P^{\nabla})^1_0(\gamma) \mid \gamma \in \Omega_M(x)\}.$$

For every $x \in M$, $H^{\nabla}|_x$ is a subgroup of $\mathrm{GL}(T_xM)$. If M is connected, it is well-known that, for any two points $x, y \in M$, there exists an absolutely continuous curve $\gamma : [0,1] \to M$ joined x and y. Then, $(P^{\nabla})_0^1(\gamma) H^{\nabla}|_x (P^{\nabla})_1^0(\gamma) = H^{\nabla}|_y$ and hence, $H^{\nabla}|_x$ and $H^{\nabla}|_y$ are conjugate subgroups of $\mathrm{GL}(T_xM)$. In other words, one can define $H^{\nabla} \subset \mathrm{GL}(n)$ the holonomy group of the linear connection ∇ (cf. [23]). Additionally, let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection associated with g, then $H^{\nabla}|_x$ is a subgroup of $\mathrm{O}(T_xM)$. If (M, g) is oriented, then $H^{\nabla}|_x$ is a subgroup of $\mathrm{SO}(T_xM)$. In this context, let F be an orthonormal frame of M at x, we write

$$H^{\nabla}|_F = \{\mathcal{M}_{F,F}(A) \mid A \in H^{\nabla}|_x\}.$$

This is a subgroup of $\mathrm{SO}(n)$, isomorphic (as Lie group) to $H^{\nabla}|_x$. The Lie algebra of the holonomy group $H^{\nabla}|_x$ (resp. $H^{\nabla}|_F$) will be denoted by $\mathfrak{h}^{\nabla}|_x$ (resp. $\mathfrak{h}^{\nabla}|_F$). Then $\mathfrak{h}^{\nabla}|_x$ is a Lie subalgebra of the Lie algebra $\mathfrak{so}(T_xM)$ of g-antisymmetric linear maps $T_xM \to T_xM$ and $\mathfrak{h}^{\nabla}|_F$ is a Lie subalgebra of $\mathfrak{so}(n)$.

We define for any two integers $k, m \in \mathbb{N}$ the set of all linear map $\mathbb{R}^k \to \mathbb{R}^m$ by $\mathcal{L}_k(\mathbb{R}^m)$. We define $O_k(\mathbb{R}^m)$ to be the set of $B \in \mathcal{L}_k(\mathbb{R}^m)$ satisfying

(i) $\| Bu \|_{\mathbb{R}^m} = \| u \|_{\mathbb{R}^k}$ for all $u \in \mathbb{R}^k$, if $k \leq m$;

(ii) B is surjective and $\| Bu \|_{\mathbb{R}^m} = \| u \|_{\mathbb{R}^k}$ for all $u \in (kerB)^{\perp}$ (where $S^{\perp}$ is the orthogonal complement of $S \subset \mathbb{R}^k$ with respect to $\langle .,. \rangle_{\mathbb{R}^k}$), if $k \geq m$.

In the statement of the above definition, we have the next results.

1. $O_k(\mathbb{R}^m)$ is a smooth closed submanifold of $\mathcal{L}_k(\mathbb{R}^m)$.
2. The restriction of the morphism $\mathcal{L}_k(\mathbb{R}^m) \to \mathcal{L}_m(\mathbb{R}^k)$; $A \mapsto A^{\overline{T}}$ on $O_k(\mathbb{R}^m) \to O_m(\mathbb{R}^k)$ is a diffeomorphism with $A^{\overline{T}}$ is the matrix transpose of A,
3. If $k \neq m$, then $O_k(\mathbb{R}^m)$ is connected. If $k = m$, $O_k(\mathbb{R}^k)$ is diffeomorphic to $\mathrm{O}(k)$, the set of $k \times k$ orthogonal matrices.

We can envisage the previous issue on an n-dimensional Riemannian manifold (M, g). Thus, for any $x \in M$ and any $k \in \mathbb{N}$, we define $\mathcal{L}_k(M)|_x$ to be the space of all linear maps $\mathbb{R}^k \to T_xM$. Set $\mathcal{L}_k(M) := \bigcup_{x \in M} \mathcal{L}_k(M)|_x$. We state the following definition and proposition.

Definition 2.2.2. *We define the subset $O_k(M)$ of $\mathcal{L}_k(M)$ of all elements $B \in \mathcal{L}_k(M)|_x$, $x \in M$, such that*

(i) if $1 \leq k \leq dimM$, we have that $\| Bu \|_g = \| u \|_{\mathbb{R}^k}$ for all $u \in \mathbb{R}^k$;

(ii) if $k \geq dimM$, we have B is surjective and $\| Bu \|_g = \| u \|_{\mathbb{R}^k}$ for all $u \in (kerB)^{\perp}$ (where $\| \, . \, \|_{\mathbb{R}^k}$ is the Euclidean norm in $\mathbb{R}^k$ and $\perp$ is taken with respect to the Euclidean inner product in $\mathbb{R}^k$).

Consider the map $\pi_{\mathcal{L}_k(M)} : \mathcal{L}_k(M) \to M$ defined by $B \mapsto x$ for $B \in \mathcal{L}_k(M)|_x$ and $\pi_{O_k(M)} := \pi_{\mathcal{L}_k(M)}|_{O_k(M)} : O_k(M) \to M$. We have the standard followings results.

Proposition 2.2.3. *1. For every $k \in \mathbb{N}$, the map $\pi_{\mathcal{L}_k(M)}$ is a smooth vector bundle over M, isomorphic to the direct sum bundle $\bigoplus_{i=1}^k TM \to M$.*

2. For all $k \in \mathbb{N}$, the map $\pi_{O_k(M)}$ defines a smooth sub-bundle of $\pi_{\mathcal{L}_k(M)}$ whose typical fiber is $O_k(\mathbb{R}^n)$, where $\mathbb{R}^n$ is equipped with the Euclidean metric.

3. If (M, g) is a connected Riemannian manifold of dimension n and if $k \neq n$ for any $k \in \mathbb{N}$ then $O_k(M)$ is connected.

4. If (M, g) be a connected Riemannian manifold of dimension n and if M is not orientable, then $O_n(M)$ is connected.

(For more details and the proof, see Section 9 in [25]).

Along the chapters, we use $(\overline{M}, \overline{g})$ to denote $(M, g) \times (\hat{M}, \hat{g})$, the Riemannian product manifold of M and $\hat{M}$, endowed with the product metric $\overline{g} := g \oplus \hat{g}$. Similarly, ∇, $\hat{\nabla}$, $\overline{\nabla}$ (resp. R, $\hat{R}$, $\overline{R}$) represent the Levi-Civita connections (resp. the Riemannian curvature tensors) of (M, g), $(\hat{M}, \hat{g})$, $(\overline{M}, \overline{g})$, respectively. By Koszul formula, we have

$$\overline{\nabla}_{(X,\hat{X})}(Y, \hat{Y}) = (\nabla_X Y, \hat{\nabla}_{\hat{X}} \hat{Y}), \quad \forall X,\ Y \in VF(M), \quad \forall \hat{X},\ \hat{Y} \in VF(\hat{M}),$$

and,

$$\overline{R}((X, \hat{X}), (Y, \hat{Y}))(Z, \hat{Z}) = (R(X, Y)Z, \hat{R}(\hat{X}, \hat{Y})\hat{Z}), \quad \forall X,\ Y,\ Z \in T_x M, \quad \forall \hat{X},\ \hat{Y},\ \hat{Z} \in T_{\hat{x}} \hat{M}.$$

The aim of last definitions is to give a short review on the definitions of homogeneous group and Carnot group (see Chapter 1 and Chapter 2 in [5]).

Definition 2.2.4. *Let $(\mathbb{R}^N, \star)$ be a Lie group, then we say that $(\mathbb{R}^N, \star)$ is an homogeneous Lie group if there exists an N-tuple of real numbers $\sigma = (\sigma_1, \ldots, \sigma_N)$ which verified $1 \leq \sigma_1 \leq \cdots \leq \sigma_N$ and such that the dilation*

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N; \quad \delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N)$$

is a group automorphism for all $\lambda > 0$. We denote this homogeneous Lie group by $(\mathbb{R}^N, \circ, \delta_\lambda)$.

we recall that if $\mathfrak{g}$ is the Lie algebra of the Lie group $(\mathbb{R}^N, \star)$, then there exists an isomorphism of vector spaces from $\mathbb{R}^N$ onto $\mathfrak{g}$. Let Z_i be the image of the canonical basis $\frac{\partial}{\partial x_i}$ of $\mathbb{R}^N$ at the origin by this isomorphism, for $i = 1, \ldots, N$. Z_i, $i = 1, \ldots, N$, is called the Jacobian basis of $\mathfrak{g}$.

Definition 2.2.5. *Let $(\mathbb{R}^N, \star)$ a Lie group, we say that $(\mathbb{R}^N, \star)$ is an homogeneous Carnot group if the two conditions below are satisfied*

C_1. $\mathbb{R}^N$ split as $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ and the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$,

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \ldots, x_{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)} \ldots, \lambda^r x^{(r)}), \ \textit{with } x^{(i)} \in \mathbb{R}^{N_i},$$

is a group automorphism for all $\lambda > 0$.

C_2. The vectors Z_i, for $i = 1, \ldots, N_1$, of the Jacobian basis of $\mathfrak{g}$ verifies

$$Lie(Z_1, \ldots, Z_{N_1}) = \mathfrak{g},$$

where $Lie(Z_1, \ldots, Z_{N_1})$ is the Lie algebra generated by the vectors Z_i, $i = 1, \ldots, N_1$.

Moreover, if $W^{(k)}$ is the vector space spanned by the Lie brackets of length k of $Z_1, \ldots, Z_{N_1}$, i.e.,

$$W^{(k)} := span\{[Z_{j_1}, \ldots, [Z_{j_{k-1}}, Z_{j_k}] \ldots]|_x, (j_1, \ldots, j_k) \in \{1, \ldots, N_1\}^k\},$$

then,

$$\mathfrak{g} = W^{(1)} \oplus \ldots . \oplus W^{(r)}$$

and

$$[W^{(1)}, W^{(i-1)}]|_x = W^{(i)}, 2 \leq i \leq r, \quad \textit{and}, \quad [W^{(1)}, W^{(r)}]|_x = 0.$$

We say that $(\mathbb{R}^N, \star)$ is an homogeneous Carnot group of step r and N_1 generators.

Chapter 3

Rolling and Development Manifolds

In this chapter, we continue the study initiated in [11] of the rolling of two smooth connected complete oriented Riemannian manifolds (M, g) and $(\hat{M}, \hat{g})$ of dimensions n and $\hat{n}$ respectively, where the integers n and $\hat{n}$ are now not necessarily equal. Two sets of constraints are usually considered, namely the rolling without spinning on the one hand and the rolling without spinning nor spinning on the other hand. Two distributions of dimensions $(n + \hat{n})$ and n are then associated to the driftless control affine systems $(\Sigma)_{NS}$ and $(\Sigma)_R$ respectively. This generalizes the rolling problems considered in [11] where both manifolds had the same dimension. We also define the development of affine manifolds of same dimensions. We then introduce the rolling development notion and their lifts and distributions.

3.1 Statement of Rolling Motion

3.1.1 The State Space Q

Definition 3.1.1. *Let (M, g) and $(\hat{M}, \hat{g})$ be two Riemannian manifolds of dimensions n and $\hat{n}$ respectively. The state space $Q = Q(M, \hat{M})$ for the problem of rolling of M against $\hat{M}$ considered below is defined as follows:*

(i) if $n \leq \hat{n}$,

$$Q(M, \hat{M}) := \{A \in T^*M \otimes T\hat{M} \mid \hat{g}(AX, AY) = g(X, Y),\ X, Y \in T_xM, x \in M\}.$$

(ii) if $n \geq \hat{n}$,

$$\begin{aligned} Q(M, \hat{M}) := \ & \{A \in T^*M \otimes T\hat{M} \mid \hat{g}(AX, AY) = g(X, Y),\ X, Y \in (\ker A)^{\perp}, \\ & \qquad A \text{ is onto a tangent space of } \hat{M}\}. \end{aligned}$$

Writing $A^T : T_{\hat{x}}\hat{M} \to T_xM$ the $(g, \hat{g})$-transpose of A, we have that $(\ker A)^{\perp} = \mathrm{im}(A^T)$, and evidently $A^T A = \mathrm{id}_{T_x M}$ if $n \leq \hat{n}$ and $AA^T = \mathrm{id}_{T_{\hat{x}}\hat{M}}$ if $n \geq \hat{n}$. Also, define

$$\begin{aligned} \pi_{Q(M,\hat{M}),M\times\hat{M}} &:= \pi_{T^*M\otimes T\hat{M},M\times\hat{M}}|_{Q(M,\hat{M})} : Q(M,\hat{M}) \to M \times \hat{M}, \\ \pi_{Q(M,\hat{M}),M} &:= \pi_{T^*M\otimes T\hat{M},M}|_{Q(M,\hat{M})} : Q(M,\hat{M}) \to M, \\ \pi_{Q(M,\hat{M}),\hat{M}} &:= \pi_{T^*M\otimes T\hat{M},\hat{M}}|_{Q(M,\hat{M})} : Q(M,\hat{M}) \to \hat{M}. \end{aligned} \tag{3.1}$$

where $\pi_{T^*M\otimes T\hat{M},M\times\hat{M}}$ is the smooth vector subbundle of the bundle of (1,1)-tensors $\pi_{T^1_1(M\times\hat{M})}$ on $M\times\hat{M}$. Thus $\pi_{T^*M\otimes T\hat{M},M}$ and $\pi_{T^*M\otimes T\hat{M},\hat{M}}$ are the projections of $\pi_{T^*M\otimes T\hat{M},M\times\hat{M}}$ onto M and $\hat{M}$ respectively. For any $q \in Q(M,\hat{M})$, we use the notation $q = (x, \hat{x}; A)$ where $x = \pi_{Q(M,\hat{M}),M}(q)$ and $\hat{x} = \pi_{Q(M,\hat{M}),\hat{M}}(q)$.

Proposition 3.1.2. *(i) The space $Q(M,\hat{M})$ is a smooth closed submanifold of $T^*M\otimes T\hat{M}$ of dimension:*

$$dim(Q) = n + \hat{n} + n\hat{n} - \frac{N(N+1)}{2}, \text{ where } N := min\{n, \hat{n}\},$$

and $\pi_{Q(M,\hat{M}),M}$ is a smooth subbundle of $\pi_{T^*M\otimes T\hat{M},M}$ with typical fiber $O_n(\hat{M})$.

(ii) The map

$$\tau_{M,\hat{M}} : T^*M \otimes T\hat{M} \to T^*\hat{M} \otimes TM; \quad (x, \hat{x}; A) \mapsto (\hat{x}, x; A^T),$$

is a diffeomorphism and its restriction to $Q(M,\hat{M})$gives the diffeomorphism

$$\overline{T} : Q(M,\hat{M}) \to Q(\hat{M}, M) =: \hat{Q}; \quad \overline{T}(x, \hat{x}; A) = \tau_{M,\hat{M}}|_Q(x, \hat{x}; A) = (\hat{x}, x; A^T). \tag{3.2}$$

(iii) If $n \neq \hat{n}$ or if one of M and $\hat{M}$ is not orientable, then the space $Q(M,\hat{M})$ is connected.

Proof. (*i*) It is clearly enough to prove the result only for $n \leq \hat{n}$. In that case, the (vertical) fiber of Q is isomorphic to the grasmannian of n-dimensional planes in an $\hat{n}$-dimensional euclidean space, hence the result.

(*ii*) First at all, we see that $\tau_{\hat{M},M}$ is the inverse map of $\tau_{M,\hat{M}}$, thus $\tau_{M,\hat{M}}$ is a diffeomorphism. Moreover, one has $(\hat{x}, x; A^T) \in \hat{Q}$ for every $(x, \hat{x}; A) \in Q$. Indeed, according to (i), one may assume that $n \leq \hat{n}$. Let $\hat{X}, \hat{Y} \in (\ker A^T)^\perp$. Since $(\ker A^T)^\perp = \mathrm{im}(A)$, there are $X, Y \in T_x M$ such that $AX = \hat{X}, AY = \hat{Y}$, and because $A^T A = \mathrm{id}_{T_x M}$, we get that $g(A^T\hat{X}, A^T\hat{Y}) = g(X, Y) = \hat{g}(AX, AY) = \hat{g}(\hat{X}, \hat{Y})$. Now, take the map

$$\overline{S} : \hat{Q} \to Q; \quad \overline{S}(\hat{x}, x; B) = (x, \hat{x}; B^T),$$

which is well-defined because $(\ker B)^\perp = \mathrm{im}(B^T)$ and $BB^T = \mathrm{id}_{T_x M}$. Thus, for all $X, Y \in T_x M$, one obtains $\hat{g}(B^T X, B^T Y) = g(BB^T X, BB^T Y) = g(X, Y)$. Therefore, $\overline{T}$ and $\overline{S}$ are smooth inverse maps to each other.

(iii) This follows from Proposition 1.2.2. ☐

Corollary 3.1.3. *The map $\pi_{Q(M,\hat{M})} : Q(M,\hat{M}) \to M \times \hat{M}$ is a bundle whose typical fiber is diffeomorphic to $O_n(\mathbb{R}^{\hat{n}})$.*

Proof. For a given point $(x_0, \hat{x}_0) \in M \times \hat{M}$, take any g-orthonormal (resp. $\hat{g}$-orthonormal) frame $F = (X_1, ..., X_n)$ (resp. $\hat{F} = (\hat{X}_1, ..., \hat{X}_{\hat{n}})$) defined on some open neighbourhood U of x_0 (resp. $\hat{U}$ of $\hat{x}_0$). Fix a $q = (x, \hat{x}; A) \in (\pi_{Q(M,\hat{M})})^{-1}(U \times \hat{U})$, define $G_{F,\hat{F}}(A)$ to be the $\hat{n} \times n$-matrix whose element on the i-th row, j-th column is $\hat{g}(\hat{X}_i|_{\hat{x}}, AX_j|_x)$ and set

$$\tau_{F,\hat{F}} : (\pi_{Q(M,\hat{M})})^{-1}(U \times \hat{U}) \to (U \times \hat{U}) \times (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}}; \quad \tau_{F,\hat{F}}(x, \hat{x}; A) = ((x, \hat{x}), G_{F,\hat{F}}(A)).$$

Using Proposition 3.1.2, it is easy to see that $\tau_{F,\hat{F}}$ is smooth, injective and its image is $(U\times\hat{U})\times O_n(\mathbb{R}^{\hat{n}})$. Moreover, its inverse map $\tau^{-1}_{F,\hat{F}} : (U\times\hat{U})\times O_n(\mathbb{R}^{\hat{n}}) \to (\pi_{Q(M,\hat{M})})^{-1}(U\times\hat{U})$ is given by

$$\tau^{-1}_{F,\hat{F}}((x, \hat{x}), B) = (x, \hat{x}; \sum_{j=1}^{n}\sum_{i=1}^{\hat{n}} B_{ij}\, g(\cdot, X_j|_x)\hat{X}_i),$$

where B_{ij} is the element on i-th row, j-th column of B. The fact that $\tau_{F,\hat{F}}$ and $\tau^{-1}_{F,\hat{F}}$ are smooth is easily established. ☐

Proposition 3.1.4. *Let $q = (x, \hat{x}; A) \in Q$ and $B \in T^*_x M \otimes T_{\hat{x}}\hat{M}$. Then $\nu(B)|_q$ is tangent to Q (i.e. is an element of $V|_q(\pi_Q)$) if and only if*

(i) $A^T B \in \mathfrak{so}(T_x M)$, if $n \le \hat{n}$.

(ii) $BA^T \in \mathfrak{so}(T_{\hat{x}}\hat{M})$, if $n \ge \hat{n}$.

Proof. Note that the set of $B \in T^*_x M \otimes T_{\hat{x}}\hat{M}$ such that $A^T B \in \mathfrak{so}(T_x M)$ and the set of $B \in T^*_x M \otimes T_{\hat{x}}\hat{M}$ such that $BA^T \in \mathfrak{so}(T_{\hat{x}}\hat{M})$ both have dimension equal to dim $\pi_Q^{-1}(x, \hat{x})$. Therefore, it is sufficient to show that $V|_q(\pi_Q) \subseteq \mathfrak{so}(T_x M)$ when $n \le \hat{n}$ and $V|_q(\pi_Q) \subseteq \mathfrak{so}(T_{\hat{x}}\hat{M})$ when $n \ge \hat{n}$. We only prove Item (i) since the other follows by using Eq. (3.2). If $n \le \hat{n}$ and $X \in T_x M$, then $A^T AX = X$. For any $B \in T^*_x M \otimes T_{\hat{x}}\hat{M}$ tangent to Q, we have $\nu(B)|_q X = 0$. Then, $0 = \nu(B)|_q(.)^T(.)X = B^T AX + A^T BX$ and hence $B^T A + A^T B = 0$ because X was arbitrary. Same analysis as (i): if $n \ge \hat{n}$ and $\hat{X} \in T_{\hat{x}}\hat{M}$, then we have $AA^T\hat{X} = \hat{X}$. For any $B \in T^*_x M \otimes T_{\hat{x}}\hat{M}$ tangent to Q, we have $\nu(B)|_q\hat{X} = 0$. Then, $0 = \nu(B)|_q(.)(.)^T\hat{X} = BA^T\hat{X} + AB^T\hat{X}$ and hence the conclusion. ☐

3.1.2 The Rolling Lifts and Distributions

Since we are interested in the rolling motion without spinning nor slipping, we formulate these conditions by taking an absolutely continuous curve on Q, $q : [a, b] \to Q$; $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ and making the following definitions.

Definition 3.1.5. *The curve $q(\cdot)$ is said to describe:*

(i) A rolling motion without spinning of M against $\hat{M}$ if:

$$\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))} A(t) = 0 \text{ for a.e. } t \in [a, b]. \tag{3.3}$$

(ii) A rolling motion without slipping of M against $\hat{M}$ if we have:

$$A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t) \text{ for a.e. } t \in [a, b]. \tag{3.4}$$

(iii) A rolling motion without slipping nor spinning of M against $\hat{M}$ if both conditions (i) and (ii) hold true.

By Item (iii) above, we get that the curves q of Q describing the rolling motion without slipping and spinning of M against $\hat{M}$ are exactly the integral curves of the following driftless control affine system

$$(\Sigma)_R \begin{cases} \dot{\gamma}(t) & = & u(t), \\ \dot{\hat{\gamma}}(t) & = & A(t)u(t), \\ \overline{\nabla}_{(u(t),A(t)u(t))} A(t) & = & 0, \end{cases} \qquad \text{for a.e. } t \in [a, b], \tag{3.5}$$

where the control u is a measurable TM-valued function defined on some finite interval $I \subset \mathbb{R}$.

Proposition 3.1.6. *Let A_0 be a (1,1)-tensor on $M \times \hat{M}$ (i.e. $\in T^1_1{}_{(x_0,\hat{x}_0)}(M \times \hat{M})$ for $(x_0, \hat{x}_0) \in M \times \hat{M}$) and $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ be an absolutely continuous curve in $T^*M \otimes T\hat{M}$ defined on some real interval $I \ni 0$ and satisfying (3.3). Then we have, for all $t \in I$,*

$$A(t) = P_0^t(\hat{\gamma}) \circ A(0) \circ P_t^0(\gamma),$$
$$A(0) \in Q \implies A(t) \in Q.$$

Proof. For the first implication, define $B(t) := P_0^t(\hat{\gamma}) \circ A(0) \circ P_t^0(\gamma)$. Evidently $B(0) = A(0)$, and if $X(t)$ is an arbitrary vector field along $\gamma(t)$, we have that $B(t)X(t)$ is a vector field along $\hat{\gamma}(t)$, and

$$\big(\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))} B(t)\big)X(t) + B(t)\nabla_{\dot{\gamma}(t)} X(t) = \hat{\nabla}_{\dot{\hat{\gamma}}(t)}(B(t)X(t)) = \hat{\nabla}_{\dot{\hat{\gamma}}(t)}\Big(P_0^t(\hat{\gamma})\big(A(0)P_t^0(\gamma)X(t)\big)\Big)$$
$$= P_0^t(\hat{\gamma})\frac{d}{dt}\big(A(0)P_t^0(\gamma)X(t)\big) = (P_0^t(\hat{\gamma}) \circ A(0) \circ P_t^0(\gamma))\big(\nabla_{\dot{\gamma}(t)} X(t)\big) = B(t)\nabla_{\dot{\gamma}(t)} X(t),$$

which, since $X(t)$ was arbitrary, would mean that $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}B(t) = 0$. By the basic uniqueness result for the first order ODEs, we thus have $A(t) = B(t)$ for all $t \in I$.

For the second implication, let $Y \in T_{\gamma(0)}M$, $\hat{Y} \in T_{\hat{\gamma}(0)}\hat{M}$ and set $Y(\cdot)$, $\hat{Y}(\cdot)$ the parallel transports of $Y, \hat{Y}$ along $\gamma(.)$ and $\hat{\gamma}(.)$ respectively. Next, suppose that $A(0) \in Q|_{(\gamma(0),\hat{\gamma}(0))}$ and denote $A(t) = P_0^t(\hat{\gamma}) \circ A(0) \circ P_t^0(\gamma)$. Then $A(0) \in T^*M \otimes T\hat{M}$, but from the first implication we obtain $A(t) \in T^*M \otimes T\hat{M}$ for all $t \in I$. So $A(t)Y(t) \in T_{\hat{\gamma}(t)}\hat{M}$ and thus,

$$\frac{d}{dt}\|A(t)Y(t)\|_{\hat{g}}^2 = 2\hat{g}((\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A(.))Y(t) + A(t)(\nabla_{\dot{\gamma}(t)}Y(.)), A(t)Y(t)) = 0.$$

If $n \le \hat{n}$, the initial condition for the preceding term is $\|A(0)Y(0)\|_{\hat{g}}^2 = \|A(0)Y\|_{\hat{g}}^2 = \|Y\|_g^2$. On the other hand, $\frac{d}{dt}\|Y(t)\|_g^2 = 0$ and the initial condition is $\|Y(0)\|_g^2 = \|Y\|_g^2$. So, $\|A(t)Y(t)\|_{\hat{g}}^2 = \|Y(t)\|_g^2$. Since the parallel transport $P_0^t(\gamma) : T_{\gamma(0)}M \to T_{\gamma(t)}M$ is a linear isometric isomorphism for every t, this proves $\hat{g}(A(t)X, A(t)Y) = g(X,Y)$ for every X, $Y \in T_{\gamma(t)}M$. If $n \ge \hat{n}$, we are able to repeat the previous method due to the fact $Y(t) \in (\ker A(t))^\perp$ if and only if $Y \in (\ker A(0))^\perp$.

□

Definition 3.1.7. *(i) Given $q = (x,\hat{x};A) \in T^*M \otimes T\hat{M}$ and $X \in T_xM$, $\hat{X} \in T_{\hat{x}}\hat{M}$, one defines the no-spinning lift of $(X,\hat{X})$ to be the unique vector $\mathscr{L}_{NS}(X,\hat{X})|_q$ of $T_x^*M \otimes T_{\hat{x}}\hat{M}$ at q given by*

$$\mathscr{L}_{NS}(X,\hat{X})|_q = \frac{d}{dt}\Big|_0 P_0^t(\hat{\gamma}) \circ A \circ P_t^0(\gamma) \quad \big(\in T_q(T^*M \otimes T\hat{M})\big),$$

where γ (resp. $\hat{\gamma}$) is any smooth curves on M (resp. $\hat{M}$) such that $\gamma(0) = x$, $\dot{\gamma}(0) = X$ (resp. $\hat{\gamma}(0) = \hat{x}$, $\dot{\hat{\gamma}}(0) = \hat{X}$).

Moreover, if $X, \hat{X}$ are (locally defined) vector fields on $M, \hat{M}$, respectively, one writes $\mathscr{L}_{NS}(X,\hat{X})$ for the (locally defined) vector field on $T^*M \otimes T\hat{M}$ whose value at q is $\mathscr{L}_{NS}(X,\hat{X})|_q$.

(ii) No-Spinning distribution $\mathscr{D}_{NS}$ on $T^*M \otimes T\hat{M}$ is an $(n+\hat{n})$-dimensional smooth distribution, whose plane at $q = (x,\hat{x};A) \in T^*M \otimes T\hat{M}$ is defined by

$$\mathscr{D}_{NS}|_q = \mathscr{L}_{NS}(T_{(x,\hat{x})}(M \times \hat{M}))|_q.$$

By Proposition 3.1.6, $\mathscr{L}_{NS}$ can be restricted to Q so that

$$\mathscr{L}_{NS}(X,\hat{X})|_q \in T_qQ, \quad \mathscr{D}_{NS}|_q \subset T_qQ,$$

for any $q \in Q$ and $X \in T_xM$, $\hat{X} \in T_{\hat{x}}\hat{M}$ as in the definition above.

Hence, we have $\mathscr{D}_{NS}|_Q$ is an $(n+\hat{n})$-dimensional (smooth) distribution on Q, which we also write as $\mathscr{D}_{NS}$ in the sequel. The next proposition gathers basic properties of $\mathscr{D}_{NS}$.

Proposition 3.1.8. *1. $(\pi_{T^*M\otimes T\hat{M}})_*$ (resp. $(\pi_Q)_*$) maps $\mathscr{D}_{NS}|_q$ isomorphically onto $T_{(x,\hat{x})}(M \times \hat{M})$ for every $q = (x,\hat{x};A) \in T^*M \otimes T\hat{M}$ (resp. $\in Q$).*

2. *If $\overline{X} \in T_{(x,\hat{x})}(M \times \hat{M})$, A is a local section of $\pi_{T^*M\otimes T\hat{M}}$ and A_* its push-forward, then we have:*
$$\mathscr{L}_{NS}(\overline{X})|_{A(x,\hat{x})} = A_*(\overline{X}) - \nu(\overline{\nabla}_{\overline{X}}A)|_{A(x,\hat{x})}. \tag{3.6}$$

3. *An absolutely continuous curve $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ on $T^*M \otimes T\hat{M}$ or Q is tangent to $\mathscr{D}_{NS}$ for a.e. t if and only if $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A = 0$ for a.e. t.*

Recall that $\overline{\nabla}$ is the product (Levi-Civita) connection on $\overline{M} = M \times \hat{M}$.

Proof. The proofs of parts 1. and 2. follow that of Proposition 3.20 and Proposition 3.22 of Section 3 in [10]. Part 3. is a consequence of Eq. (3.6) so that

$$\mathscr{L}_{NS}(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))|_{q(t)} = \dot{A}(t) - \nu(\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A)|_{q(t)}.$$

□

Remark 3.1.9. In the previous proposition, the two terms on the right side of Eq. (3.6) are separately elements of $T_q(T^*M \otimes T\hat{M})$, but their difference belongs to T_qQ. Moreover, this equation indicates the decomposition of the map A_* with respect to the two direct sum decompositions:

$$\begin{aligned} T(T^*M \otimes T\hat{M}) =& \mathscr{D}_{NS} \oplus_{T^*M\otimes T\hat{M}} V(\pi_{T^*M\otimes T\hat{M}}), \\ TQ =& \mathscr{D}_{NS} \oplus_Q V(\pi_Q). \end{aligned}$$

We shall now define a subdistribution $\mathscr{D}_R$ of $\mathscr{D}_{NS}$ which has the property that tangent curves to $\mathscr{D}_R$ are exactly those curves in Q (or $T^*M \otimes T\hat{M}$) that verify both the no-slipping and no-spinning conditions, i.e., are the curves modelled by the system $\Sigma_{(R)}$.

Definition 3.1.10. *(i) For any $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, the rolling lift of $X \in T_xM$ is the vector $\mathscr{L}_R(X)|_q$ of $T^*M \otimes T\hat{M}$ at q defined by*

$$\mathscr{L}_R(X)|_q := \mathscr{L}_{NS}(X, AX)|_q. \tag{3.7}$$

Moreover, if X is a (locally defined) vector field on M, one writes $\mathscr{L}_{NS}(X)$ for the (locally defined) vector field on $T^*M \otimes T\hat{M}$ whose value at q is $\mathscr{L}_{NS}(X)|_q$.

(ii) The Rolling distribution $\mathscr{D}_R$ on $T^*M \otimes T\hat{M}$ is the n-dimensional smooth distribution whose plane at every $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ is given by

$$\mathscr{D}_R|_q := \mathscr{L}_R(T_xM)|_q. \tag{3.8}$$

Like right below the definition 3.1.7, one can restrict $\mathscr{L}_R$ to Q such that

$$\mathscr{L}_R(X)|_q \in T_qQ, \quad \mathscr{D}_R|_q \subset T_qQ,$$

for all $q = (x, \hat{x}; A) \in Q$ and $X \in T_xM$.

Corollary 3.1.11. *(i) $(\pi_{Q,M})_*$ maps $\mathscr{D}_R|_q$ isomorphically onto T_xM for for every $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ (resp. $q \in Q$).*

(ii) An absolutely continuous curve $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ on $T^*M \otimes T\hat{M}$ (resp. on Q) is a rolling curve if and only if it is tangent to $\mathscr{D}_R$ for a.e. t i.e. if and only if $\dot{q}(t) = \mathscr{L}_R(\dot{\gamma}(t))|_{q(t)}$ for a.e. t.

While some of the results that follow hold true in both spaces Q and $T^*M \otimes T\hat{M}$, we mainly focus on Q, which is the state space of primary interest for the purposes of rolling. The generalization of such a result to $T^*M \otimes T\hat{M}$, if it makes sense there, is usually transparent, and, if need be, we will use such generalizations without further mention for convenience in some of the forthcoming proof.

We have the following fundamental result whose proof follows the same lines as that of Proposition 3.27 of Section 3 in [10].

Proposition 3.1.12. *(i) For every $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and every absolutely continuous $\gamma : [0, a] \to M$, $a > 0$, such that $\gamma(0) = x_0$, there exists a unique absolutely continuous $q : [0, a'] \to Q$, $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, with $0 < a' \leq a$ which is tangent to $\mathscr{D}_R$ a.e. and $q(0) = q_0$. We denote this unique curve q by*

$$t \mapsto q_{\mathscr{D}_R}(\gamma, q_0)(t) = (\gamma(t), \hat{\gamma}_{\mathscr{D}_R}(\gamma, q_0)(t); A_{\mathscr{D}_R}(\gamma, q_0)(t)), \tag{3.9}$$

and refer to it as the rolling curve with initial conditions (γ, q_0), or along γ with initial position q_0.

(ii) Moreover, if $(\hat{M}, \hat{g})$ is a complete manifold, one can choose $a' = a$ above.

(iii) Conversely, any absolutely continuous curve $q : [0, a] \mapsto Q$ tangent to $\mathscr{D}_R$ a.e. has the form $q_{\mathscr{D}_R}(\gamma, q(0))$ where $\gamma = \pi_{Q,M} \circ q$.

Remark 3.1.13. Let (N, h) be a Riemannian manifold and $y_0 \in N$, we define a bijection $\Lambda^{\nabla^h}_{y_0}(\cdot)$ from the set of absolutely continuous curves $\gamma : [0, 1] \to N$ starting at y_0 onto an open subset of the Banach space of absolutely continuous curves $[0, 1] \to T_{y_0}N$ starting at 0, by

$$\Lambda^{\nabla^h}_{y_0}(\gamma)(t) = \int_0^t (P^{\nabla^h})^0_s(\gamma)\dot{\gamma}(s)ds \ \in T_{y_0}N, \ \forall t \in [0, 1].$$

It follows from Proposition 3.1.12 that the rolling curve with initial conditions (γ, q_0) is given by:

$$q_{\mathscr{D}_R}(\gamma, q_0)(t) = (\gamma(t), \hat{\Lambda}^{-1}_{\hat{x}_0}(A_0 \circ \Lambda_{x_0}(\gamma))(t); P^t_0(\hat{\Lambda}^{-1}_{\hat{x}_0}(A_0 \circ \Lambda_{x_0}(\gamma))) \circ A_0 \circ P^0_t(\gamma)). \tag{3.10}$$

Moreover, if the curve γ is the geodesic on M given by $\gamma(t) = exp_{x_0}(tX)$ with $\gamma(0) = x_0$ and $\dot{\gamma}(0) = X \in T_{x_0}M$, then, for $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, the rolling curve $q_{\mathscr{D}_R}(\gamma, q_0) : [0, a'] \to Q$, $0 < a' \leq a$, is given by

$$q_{\mathscr{D}_R}(\gamma, q_0)(t) = (\gamma(t), \hat{\gamma}_{\mathscr{D}_R}(\gamma, q_0)(t) = \widehat{exp}_{\hat{x}_0}(tA_0X), A_{\mathscr{D}_R}(\gamma, q_0)(t) = P^t_0(\hat{\gamma}_{\mathscr{D}_R}(\gamma, q_0)) \circ A_0 \circ P^0_t(\gamma)).$$

We also have that if $\hat{M}$ is complete then $a = a'$.

Let $\widehat{\mathscr{L}_{NS}}$ and $\widehat{\mathscr{L}_R}$ (resp. $\widehat{\mathscr{D}_{NS}}$ and $\widehat{\mathscr{D}_R}$) be the no-spinning and rolling lifts (resp. the no-spinning and rolling distributions), respectively, on $\hat{Q} := Q(\hat{M}, M)$. Thus, $\dim \widehat{\mathscr{D}_{NS}} = n + \hat{n} = \dim \mathscr{D}_{NS}$ but, in contrary, $\dim \widehat{\mathscr{D}_R} = \hat{n}$, $\dim \mathscr{D}_R = n$. This shows that the model of rolling of manifolds of different dimensions against each other is not symmetric with respect to the order of M and $\hat{M}$.

Proposition 3.1.14. *Let $\overline{T}$ the mapping defined by* (3.2)*, we have the followings results:*

1. $\overline{T}_* \mathscr{D}_{NS} = \widehat{\mathscr{D}_{NS}}$,
2. $\overline{T}_* V(\pi_Q) = V(\pi_{\hat{Q}})$,
3. *when $n \leq \hat{n}$, we have $\overline{T}_* \mathscr{D}_R \subset \widehat{\mathscr{D}_R}$.*

Proof. We can assume, without loss of generality, that $n \leq \hat{n}$.

1. For $q_0 = (x_0, \hat{x}_0; A_0) \in Q(M, \hat{M})$, let γ, $\hat{\gamma}$ be a smooth paths in M, $\hat{M}$ starting at x_0, $\hat{x}_0$, respectively, at t = 0. We have that $(P_0^t(\hat{\gamma}) \circ A_0 \circ P_t^0(\gamma))^T = P_0^t(\gamma) \circ A_0^T \circ P_t^0(\hat{\gamma})$, then

$$\overline{T}(\gamma(t), \hat{\gamma}(t); P_0^t(\hat{\gamma}) \circ A_0 \circ P_t^0(\gamma)) = (\hat{\gamma}(t), \gamma(t); P_0^t(\gamma) \circ \overline{T}(x_0, \hat{x}_0; A_0) \circ P_t^0(\hat{\gamma})).$$

This immediately shows, by differentiating it with respect to $\frac{d}{dt}|_0$ and using the definition of $\mathscr{L}_{NS}$, that

$$\overline{T}_*|_{q_0} \mathscr{L}_{NS}(X, \hat{X})|_{q_0} = \widehat{\mathscr{L}_{NS}}(\hat{X}, X)|_{\overline{T}(q_0)},$$

where $X = \dot{\gamma}(0)$, $\hat{X} = \dot{\hat{\gamma}}(0)$. In particular, $\overline{T}_*$ maps $\mathscr{D}_{NS}$ isomorphically onto $\widehat{\mathscr{D}_{NS}}$.

2. Let $\nu(B)|_{q=(x,\hat{x};A)} \in V|_q(\pi_Q)$, B verifies $A^T B \in \mathfrak{so}(T_x M)$ then $\nu(B^T)|_{\overline{T}(q)} \in V|_{\overline{T}(q)}(\pi_{\hat{Q}})$. Then, $\overline{T}_* V(\pi_Q) = V(\pi_{\hat{Q}})$ because we have, for any $\hat{f} \in C^\infty(\hat{Q})$,

$$(\overline{T}_* \nu(B)|_q)\hat{f} = \nu(B)|_q(\hat{f} \circ \overline{T}) = \frac{d}{ds}|_0 \hat{f}(\overline{T}(x, \hat{x}; A + sB)) = \frac{d}{ds}|_0 \hat{f}(\hat{x}, x; A^T + sB^T) = \nu(B^T)|_{\overline{T}(q)}.$$

3. For $q_0 = (x_0, \hat{x}_0; A_0)$ and $X \in T_{x_0} M$, one has

$$\overline{T}_*|_{q_0} \mathscr{L}_R(X)|_{q_0} = \overline{T}_*|_{q_0} \mathscr{L}_{NS}(X, A_0 X)|_{q_0} = \widehat{\mathscr{L}_{NS}}(A_0 X, A_0^T A_0 X)|_{\overline{T}(q_0)} = \widehat{\mathscr{L}_R}(A_0 X)|_{\overline{T}(q_0)},$$

since $X = A_0^T(A_0 X) = \overline{T}(q_0)(A_0 X)$. Hence $\overline{T}$ maps $\mathscr{D}_R$ of $Q(M, \hat{M})$ into $\widehat{\mathscr{D}_R}$ of $Q(\hat{M}, M)$. ∎

3.1.3 The Lie Brackets on Q

Let $\mathcal{O}$ be an immersed submanifold of $T^*M \otimes T\hat{M}$ and write $\pi_{\mathcal{O}} := \pi_{T^*M\otimes T\hat{M}}|_{\mathcal{O}}$. If $\overline{T} : \mathcal{O} \to T^k_m(M \times \hat{M})$ with $\pi_{T^k_m(M\times\hat{M})} \circ \overline{T} = \pi_{\mathcal{O}}$ (i.e. $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^k_m(M\times\hat{M})})$) and if $q = (x, \hat{x}; A) \in \mathcal{O}$ and $\overline{X} \in T_{(x,\hat{x})}(M \times \hat{M})$ such that $\mathscr{L}_{NS}(\overline{X})|_q \in T_q\mathcal{O}$, then we want to define what it means to take the derivative $\mathscr{L}_{NS}(\overline{X})|_q\overline{T}$. Our main interest will be the case where $k = 1$, $m = 0$ i.e. $T(M \times \hat{M})$, but some arguments below require a general setting. As a first step, we take $\mathcal{O} = T^*M \otimes T\hat{M}$. We can inspire, from Eq. (3.6), the following definition

$$\mathscr{L}_{NS}(\overline{X})|_q\overline{T} := \overline{\nabla}_{\overline{X}}(\overline{T}(\tilde{A})) - \nu(\overline{\nabla}_{\overline{X}}\tilde{A})|_q\overline{T} \in T^k_m(M \times \hat{M}).$$

Here, $\overline{T}(A) = \overline{T} \circ A$ is a locally defined (k, m)-tensor field on $M \times \hat{M}$. On the other hand, if $\overline{\omega} \in \Gamma(\pi_{T^m_k(M\times\hat{M})})$ and if we write $(\overline{T}\overline{\omega})(q) := \overline{T}(q)\overline{\omega}|_{(x,\hat{x})}$ as a full contraction for $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, then we may compute

$$\begin{aligned}
(\mathscr{L}_{NS}(\overline{X})|_q\overline{T})\overline{\omega} = \;& (\overline{\nabla}_{\overline{X}}(\overline{T}(A)))\overline{\omega} - (\tfrac{d}{dt}|_0\overline{T}(A + t\overline{\nabla}_{\overline{X}}A))\overline{\omega} \\
= \;& \overline{\nabla}_{\overline{X}}(\overline{T}(A)\overline{\omega}) - \overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega} - \tfrac{d}{dt}|_0(\overline{T}(A + t\overline{\nabla}_{\overline{X}}A)\overline{\omega}) \\
= \;& \overline{\nabla}_{\overline{X}}((\overline{T}\overline{\omega})(A)) - \tfrac{d}{dt}|_0(\overline{T}\overline{\omega})(A + t\overline{\nabla}_{\overline{X}}A) - \overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega}.
\end{aligned}$$

Hence,

$$(\mathscr{L}_{NS}(\overline{X})|_q\overline{T})\overline{\omega} = \mathscr{L}_{NS}(\overline{X})|_q(\overline{T}\overline{\omega}) - \overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega}. \tag{3.11}$$

Alternatively, Eq. (3.11) represents an intrinsic definition of $\mathscr{L}_{NS}(\overline{X})|_q\overline{T}$.

Now, if $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is an immersed submanifold, we could take Eq. (3.11) as the definition of $\mathscr{L}_{NS}(\overline{X})|_q\overline{T}$ for $q \in \mathcal{O}$.

Definition 3.1.15. *Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $q = (x, \hat{x}; A) \in \mathcal{O}$ and $\overline{X} \in T_{(x,\hat{x})}(M \times \hat{M})$ be such that $\mathscr{L}_{NS}(\overline{X})|_q \in T_q\mathcal{O}$. Then for $\overline{T} : \mathcal{O} \to T^k_m(M \times \hat{M})$ such that $\pi_{T^k_m(M\times\hat{M})} \circ \overline{T} = \pi_{\mathcal{O}}$, we define $\mathscr{L}_{NS}(\overline{X})|_q\overline{T}$ to be the unique element in $T^k_m|_{(x,\hat{x})}(M \times \hat{M})$ such that Eq. (3.11) holds for every $\overline{\omega} \in \Gamma(\pi_{T^m_k(M\times\hat{M})})$ and call it the derivative of $\overline{T}$ with respect to $\mathscr{L}_{NS}(\overline{X})|_q$.*

We next present the main Lie brackets formulas obtained as in Proposition 3.45, Proposition 3.46, Proposition 3.47 of Section 3 in [10].

Proposition 3.1.16. *Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $\overline{T} = (T, \hat{T})$, $\overline{S} = (S, \hat{S}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$ be such that for all $q = (x, \hat{x}; A) \in \mathcal{O}$, $\mathscr{L}_{NS}(\overline{T}(q))|_q$, $\mathscr{L}_{NS}(\overline{S}(q))|_q \in T_q\mathcal{O}$ and U, $V \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$, be such that for all $q = (x, \hat{x}; A) \in \mathcal{O}$, $\nu(U(q))|_q$, $\nu(V(q))|_q \in T_q\mathcal{O}$. Then, one has*

1. $$\begin{aligned}
[\mathscr{L}_{NS}(\overline{T}(.)), \mathscr{L}_{NS}(\overline{S}(.))]|_q = \;& \mathscr{L}_{NS}(\mathscr{L}_{NS}(\overline{T}(q))|_q\overline{S} - \mathscr{L}_{NS}(\overline{S}(q))|_q\overline{T})|_q \\
& +\nu(AR(T(q), S(q)) - \hat{R}(\hat{T}(q), \hat{S}(q))A)|_q,
\end{aligned}$$

2. $$[\mathscr{L}_{NS}(\overline{T}(.)),\nu(U(.))]|_q = -\mathscr{L}_{NS}(\nu(U(q))|_q\overline{T})|_q + \nu(\mathscr{L}_{NS}(\overline{T}(q))|_qU)|_q,$$

3. $$[\nu(U(.)),\nu(V(.))]|_q = \nu(\nu(U(q))|_qV - \nu(V(q))|_qU)|_q.$$

Both sides of the equalities in 1. , 2. *and* 3. *are tangent to* $\mathcal{O}$.

3.2 Development of (M,∇) on $(\hat{M},\hat{\nabla})$

By Remark 3.1.13 and more precisely by (3.10), we see that one can generalize the definition of rolling curves on affine manifolds. Indeed, let M and $\hat{M}$ be smooth n-dimensional manifolds and set ∇ and $\hat{\nabla}$ to be the affine connections on M and $\hat{M}$ respectively. We recall next basic definitions and results stated in [10] and [24] on the rolling development of (M,∇) on $(\hat{M},\hat{\nabla})$.

Definition 3.2.1. *Let $\gamma:[0,1]\to M$ be an a.c. curve on M starting at $\gamma(0)=x_0$. We define the development of γ on $T_{x_0}M$ with respect to ∇ as the a.c. curve $\Lambda^{\nabla}_{x_0}(\gamma):[0,1]\to T_{x_0}M$ given by*

$$\Lambda^{\nabla}_{x_0}(\gamma)(t) = \int_0^t (P^{\nabla})^0_s(\gamma)\dot{\gamma}(s)ds, \qquad t\in[0,1].$$

Definition 3.2.2. *Let $(x_0,\hat{x}_0)\in M\times\hat{M}$, $A_0\in T^*_{x_0}M\otimes T_{\hat{x}_0}\hat{M}$ and an a.c. curve $\gamma:[0,1]\to M$ such that $\gamma(0)=x_0$. We define the development of γ onto $\hat{M}$ through A_0 with respect to $\overline{\nabla}$ as the a.c. curve $\Lambda^{\overline{\nabla}}_{A_0}(\gamma):[0,1]\to M$ given by*

$$\Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t) := (\Lambda^{\hat{\nabla}}_{\hat{x}_0})^{-1}(A_0\circ\Lambda^{\nabla}_{x_0}(\gamma))(t), \qquad t\in[0,1].$$

We also define the relative parallel transport of A_0 along γ with respect to $\overline{\nabla}$ to be the linear map

$$(P^{\overline{\nabla}})^t_0(\gamma)A_0 : T_{\gamma(t)}M \to T_{\Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t)}\hat{M},\ \textit{such that for } t\in[0,1],$$
$$(P^{\overline{\nabla}})^t_0(\gamma)A_0 := (P^{\hat{\nabla}})^t_0(\Lambda^{\overline{\nabla}}_{A_0}(\gamma))\circ A_0\circ(P^{\nabla})^0_t(\gamma).$$

Thus, we have the next definition of the rolling development of (M,∇) against $(\hat{M},\hat{\nabla})$.

Definition 3.2.3. *The state space of the rolling development of (M,∇) on $(\hat{M},\hat{\nabla})$ is*

$$Q := Q(M,\hat{M}) = \{A\in T^*_xM\otimes T_{\hat{x}}\hat{M} \mid A\in GL(n),\ x\in M \text{ and } \hat{x}\in\hat{M}\}.$$

A point $q\in Q$ is written as $q=(x,\hat{x};A)$.

We define the No-Spinning development lift of $(X,\hat{X})\in T_{(x,\hat{x})}(M\times\hat{M})$ and the No-Spinning development distribution as follows.

Definition 3.2.4. *Let $q = (x, \hat{x}; A) \in Q$, $(X, \hat{X}) \in T_{(x,\hat{x})}(M \times \hat{M})$ and γ (resp. $\hat{\gamma}$) be an a.c. curve on M (resp. on $\hat{M}$) starting at x (resp. $\hat{x}$) with initial velocity X (resp. $\hat{X}$). The No-Spinning development lift of $(X, \hat{X})$ is the unique vector $\mathscr{L}_{NS}(X, \hat{X})|_q$ of T_qQ at $q = (x, \hat{x}; A)$ given by*

$$\mathscr{L}_{NS}(X, \hat{X})|_q := \frac{d}{dt}\Big|_0 (P^{\hat{\nabla}})^t_0(\hat{\gamma}) \circ A \circ (P^{\nabla})^0_t(\gamma).$$

The No-Spinning development distribution $\mathscr{D}_{NS}$ at $q = (x, \hat{x}; A) \in Q$ is a $2n$-dimensional smooth distribution defined by

$$\mathscr{D}_{NS}|_q := \mathscr{L}_{NS}(T_{(x,\hat{x})}M \times \hat{M})|_q.$$

The definitions of the Rolling development lift of $X \in T_xM$ and the Rolling development distribution are in the following statement.

Definition 3.2.5. *Set $q = (x, \hat{x}; A) \in Q$ and $X \in T_xM$. Let γ and $\hat{\gamma}$ be a.c. curves on M and $\hat{M}$ respectively satisfying $\gamma(0) = x$, $\hat{\gamma}(0) = \hat{x}$, $\dot{\gamma}(0) = X$ and $\dot{\hat{\gamma}}(0) = AX$. We define the Rolling development lift $\mathscr{L}_R$ at $q = (x, \hat{x}; A) \in Q$ to be the injective map from T_xM onto T_qQ, such that for every $X \in T_xM$, the vector $\mathscr{L}_R(X)|_q$ is defined by*

$$\mathscr{L}_R(X)|_q := \mathscr{L}_{NS}(X, AX)|_q = \frac{d}{dt}\Big|_0 (P^{\hat{\nabla}})^t_0(\hat{\gamma}) \circ A \circ (P^{\nabla})^0_t(\gamma).$$

The Rolling development distribution $\mathscr{D}_R$ at $q = (x, \hat{x}; A) \in Q$ is an n-dimensional smooth distribution defined by

$$\mathscr{D}_R|_q := \mathscr{L}_R(T_xM)|_q.$$

We say that an a.c. curve $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ on Q, is a rolling development curve if and only if it is tangent to $\mathscr{D}_R$ for a.e. $t \in I$, where I is a bounded interval of $\mathbb{R}$, i.e. if and only if $\dot{q}(t) = \mathscr{L}_R(\dot{\gamma}(t))|_{q(t)}$ for a.e. $t \in I$.

Proposition 3.2.6. *For any $q_0 := (x_0, \hat{x}_0; A_0) \in Q$ and any a.c. curve $\gamma : [0, 1] \to M$ starting at x_0, there exist unique a.c. curves $\hat{\gamma}(t) := \Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t)$ and $A(t) := (P^{\overline{\nabla}})^t_0(\gamma)A_0$ such that $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$ and $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A(t) = 0$, for all $t \in [0, 1]$. We refer to $t \mapsto q_{\mathscr{D}_R}(\gamma, q_0) := (\gamma(t), \hat{\gamma}(t); A(t))$ as the rolling development curve of initial condition (γ, q_0) or along γ with initial position q_0.*

We next present the main computation tools obtained in Proposition 3.7, Lemma 3.18, Proposition 3.24, Proposition 3.26, Proposition 4.1, Proposition 4.6 in [26]. Let R^{∇} and T^{∇} (resp. $\hat{R}^{\hat{\nabla}}$ and $\hat{T}^{\hat{\nabla}}$) be the curvature tensor and the torsion tensor respectively of M with respect to ∇ (resp. of $\hat{M}$ with respect to $\hat{\nabla}$).

Proposition 3.2.7. *Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $\overline{Z} = (Z, \hat{Z})$, $\overline{S} = (S, \hat{S}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ be such that for all $q = (x, \hat{x}; A) \in \mathcal{O}$, $\mathscr{L}_{NS}(\overline{Z}(q))|_q$,*

$\mathscr{L}_{NS}(\overline{S}(q))|_q \in T_q\mathcal{O}$ and $U, V \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$, be such that for all $q = (x, \hat{x}; A) \in \mathcal{O}$, $\nu(U(q))|_q, \nu(V(q))|_q \in T_q\mathcal{O}$. Then, one has

$$\mathscr{L}_{NS}(\overline{Z}(A))|_q\overline{S}(\cdot) := \overline{\nabla}_{\overline{Z}(A)}(\overline{S}(A)) - \nu(\overline{\nabla}_{\overline{Z}(A)}A)|_q\overline{S}(\cdot), \tag{3.12}$$

$$\begin{aligned}
[\mathscr{L}_{NS}(\overline{Z}(\cdot)), \mathscr{L}_{NS}(\overline{S}(\cdot))]|_q &= \mathscr{L}_{NS}(\mathscr{L}_{NS}(\overline{Z}(A))|_q\overline{S}(\cdot) - \mathscr{L}_{NS}(\overline{S}(A))|_q\overline{Z}(\cdot))|_q \\
&- \mathscr{L}_{NS}(T^\nabla(Z(q), S(q)), \hat{T}^{\hat{\nabla}}(\hat{Z}(q), \hat{S}(q)))|_q \\
&+ \nu(AR^\nabla(Z(q), S(q)) - \hat{R}^{\hat{\nabla}}(\hat{Z}(q), \hat{S}(q))A)|_q,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
[\mathscr{L}_R(Z(\cdot)), \mathscr{L}_R(S(\cdot))]|_q &= \mathscr{L}_R([Z(q), S(q)])|_q \\
&+ \mathscr{L}_{NS}(AT^\nabla(Z(q), S(q)) - \hat{T}^{\hat{\nabla}}(AZ(q), AS(q)))|_q \\
&+ \nu(AR^\nabla(Z(q), S(q)) - \hat{R}^{\hat{\nabla}}(AZ(q), AS(q))A)|_q,
\end{aligned} \tag{3.14}$$

$$[\mathscr{L}_{NS}(\overline{Z}(\cdot)), \nu(U(\cdot))]|_q = -\mathscr{L}_{NS}(\nu(U(A))|_q\overline{Z}(\cdot))|_q + \nu(\mathscr{L}_{NS}(\overline{Z}(A))|_qU(\cdot))|_q, \tag{3.15}$$

$$[\nu(U(\cdot)), \nu(V(\cdot))]|_q = \nu(\nu(U(A))|_qV - \nu(V(A))|_qU)|_q. \tag{3.16}$$

Both sides of the equalities in (3.12), (3.13), (3.14), (3.15) *and* (3.16) *are tangent to* $\mathcal{O}$.

3.3 Appendix

In this section we briefly show how one writes the control system $\Sigma_{(R)}$ in local orthonormal frames.

Let $(F_i)_{1\le i\le n}$ and $(\hat{F}_j)_{1\le j\le \hat{n}}$ be local oriented orthonormal frames on M and $\hat{M}$ respectively and let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that x_0, $\hat{x}_0$ belong to the domains of definition V and $\hat{V}$ of the frames. Let $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [0, 1]$, be a curve in Q so that $\gamma \subset V$ and $\hat{\gamma} \subset \hat{V}$. For every $t \in [0, 1]$, define the unique element $\mathcal{R}(t)$ in $\mathrm{SO}(n, \hat{n})$ verifying

$$\big(A(t)F_1|_{\gamma(t)}, \ldots, A(t)F_n|_{\gamma(t)}\big) = \big(\hat{F}_1|_{\hat{\gamma}(t)}, \ldots, \hat{F}_{\hat{n}}|_{\hat{\gamma}(t)}\big)\mathcal{R}(t)$$

Define Christoffel symbols $\Gamma \in T_x^*M \otimes \mathfrak{so}(n)$ and $\hat{\Gamma} \in T_{\hat{x}}^*\hat{M} \otimes \mathfrak{so}(\hat{n})$ by

$$\Gamma(X)_i^l = g(\nabla_X F_i, F_l), \quad \hat{\Gamma}(\hat{X})_j^k = \hat{g}(\hat{\nabla}_{\hat{X}}\hat{F}_j, \hat{F}_k),$$

with $1 \le i, k \le n$, $1 \le j, k \le \hat{n}$ and $X \in T_xM$, $\hat{X} \in T_{\hat{x}}\hat{M}$.

There are unique measurable functions $u^i : [0,1] \to \mathbb{R}$, $1 \leq i \leq n$, such that, for a.e. $t \in [0,1]$,

$$\dot{\gamma}(t) = \big(F_1|_{\gamma(t)}, \ldots, F_n|_{\gamma(t)}\big) \begin{pmatrix} u^1(t) \\ \vdots \\ u^n(t) \end{pmatrix}.$$

As one can easily verify, the conditions of no-spinning (3.3) and no-slipping (3.4) translate for $(\hat{\gamma}(t), \mathcal{R}(t)) \in \hat{M} \times \mathrm{SO}(n)$ precisely to

$$\begin{aligned} &(\text{no-slip}) \quad \dot{\hat{\gamma}}(t) = \big(\hat{F}_1|_{\hat{\gamma}(t)}, \ldots, \hat{F}_{\hat{n}}|_{\hat{\gamma}(t)}\big)\mathcal{R}(t) \begin{pmatrix} u^1(t) \\ \vdots \\ u^n(t) \end{pmatrix}, \\ &(\text{no-spin}) \quad \dot{\mathcal{R}}(t) = \mathcal{R}(t)\Gamma(\dot{\gamma}(t)) - \hat{\Gamma}(\dot{\hat{\gamma}}(t))\mathcal{R}(t), \end{aligned}$$

for a.e. $t \in [0,1]$. Moreover, the latter no-spin condition can also be written as

$$\dot{\mathcal{R}}(t) = \sum_{i=1}^{n} u^i(t)\Big(\mathcal{R}(t)\Gamma(F_i|_{\gamma(t)}) - \sum_{j=1}^{\hat{n}} \mathcal{R}_{ji}(t)\hat{\Gamma}(\hat{F}_j|_{\hat{\gamma}(t)})\mathcal{R}(t)\Big),$$

for a.e. $t \in [0,1]$, where $\mathcal{R}_{ji}(t)$ is the element at j-th row, i-th column of $\mathcal{R}(t)$. From this local form, one clearly sees that the rolling system Σ_R is a driftless control affine system.

Chapter 4

Controllability Result on Rolling Riemannian Manifolds

Recall that the rolling system is said to be completely controllable if the rolling orbit is equal to the whole state space Q for some points and hence for every points of Q. Therefore, if Q is connected, then the rolling system is non-controllable if and only if there exists a point belongs to Q which its rolling orbit is not open in Q. We address in this chapter the controllability issue for the systems rolling $(\Sigma)_{NS}$ and $(\Sigma)_R$ of rolling of Riemannian manifolds defined in Chapter 2. In particular, we must provide some basic properties for the reachable sets.

4.1 Rolling Orbits and Rolling Distributions

In this section, we first characterize the rolling orbits corresponding to the (NS) and (R) problems and then we provide specific results on $\mathscr{D}_R$-orbits in the case $|n-\hat{n}|=1$.

4.1.1 General Properties of Rolling Orbits

We collect here some basic results on the structure of the orbits and the distributions of the two rolling systems. To begin with, we completely describe the reachable sets of (NS) to the holonomy groups of the Riemannian manifolds (M,g) and $(\hat{M},\hat{g})$, which are Lie subgroups of $\mathrm{SO}(n)$ and $\mathrm{SO}(\hat{n})$.

In this setting, $H|_x$ and $\hat{H}|_{\hat{x}}$ denote $H^{\nabla}|_x$ and $H^{\hat{\nabla}}|_{\hat{x}}$ respectively(for the notations, see the section 2.2). The corresponding Lie algebras will be written as $\mathfrak{h}|_x$, $\hat{\mathfrak{h}}|_{\hat{x}}$. Following the arguments of Theorem 4.1, Corollaries 4.2 and 4.3 and Proposition 4.5 of Section 4 as well as Property 5.2 of Section 5 in [10], one gets the subsequent result.

Theorem 4.1.1. *Let $q_0=(x_0,\hat{x}_0;A_0)\in Q$. Then the part of the orbit $\mathcal{O}_{\mathscr{D}_{NS}}(q_0)$ of $\mathscr{D}_{NS}$ through q_0 that lies in the π_Q-fiber over $(x_0,\hat{x}_0)$ verifies*

$$\mathcal{O}_{\mathscr{D}_{NS}}(q_0)\cap\pi_Q^{-1}(x_0,\hat{x}_0)=\{\hat{h}\circ A_0\circ h\mid \hat{h}\in\hat{H}|_{\hat{x}_0},\ h\in H|_{x_0}\}=:\hat{H}|_{\hat{x}_0}\circ A_0\circ H|_{x_0},\quad(4.1)$$

In addition, at the tangent space level, we have

$$T_{q_0}\mathcal{O}_{\mathscr{D}_{NS}}(q_0)\cap V|_{q_0}(\pi_Q) \begin{array}{l} = \nu(\{\hat{k}\circ A_0 - A_0\circ k \mid \quad k\in\mathfrak{h}|_{x_0},\ \hat{k}\in\hat{\mathfrak{h}}|_{\hat{x}_0}\})|_{q_0} \\ =: \nu(\hat{\mathfrak{h}}|_{\hat{x}_0}\circ A_0 - A_0\circ\mathfrak{h}|_{x_0})|_{q_0}. \end{array} \tag{4.2}$$

Proposition 4.1.2. *If $\hat{M}$ is complete, then for every $q_0\in Q$, the map $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),M} := \pi_{Q,M}|_{\mathcal{O}_{\mathscr{D}_R}(q_0)} : \mathcal{O}_{\mathscr{D}_R}(q_0)\to M$ defines a smooth subbundle of $\pi_{Q,M}$.*

We next compute the first commutators of $\mathscr{L}_R(X)$ where $X\in \mathrm{VF}(M)$. The resulting formulas are obtained as in Proposition 5.9 of Section 5 in [10].

Theorem 4.1.3. *If $X,\ Y\in\mathrm{VF}(M)$, $q=(x,\hat{x};A)\in Q$, then*

$$[\mathscr{L}_R(X),\mathscr{L}_R(Y)]|_q = \mathscr{L}_R([X,Y])|_q + \nu(AR(X,Y) - \hat{R}(AX,AY)A)|_q. \tag{4.3}$$

Definition 4.1.4. *For $q=(x,\hat{x};A)\in Q$, we define the rolling curvature Rol_q at q by*

$$\mathsf{Rol}_q(X,Y) := AR(X,Y) - \hat{R}(AX,AY)A,\quad X,Y\in T_xM.$$

If $X,Y\in\mathrm{VF}(M)$, we write $\mathsf{Rol}(X,Y)$ for the map $Q\to T^*M\otimes T\hat{M}$; $q\mapsto \mathsf{Rol}_q(X,Y)$.
Similarly, for $k\geq 0$, we define the k-th covariant derivative of Rol at q by

$$(\overline{\nabla}^k\mathsf{Rol})_q(X,Y,Z_1,...,Z_k) := A(\nabla^k R)(X,Y,(.),Z_1,...,Z_k) - (\hat{\nabla}^k\hat{R})(AX,AY,A(.),AZ_1,...,AZ_k).$$

Clearly, for all $(x,\hat{x};A)\in Q$,

$$A^T\mathsf{Rol}_q(X,Y),\ A^T(\overline{\nabla}^k\mathsf{Rol})_q(X,Y,Z_1,...,Z_k)\in\mathfrak{so}(T_xM)\ if\ n\leq\hat{n},$$

and

$$\mathsf{Rol}_q(X,Y)A^T,\ (\overline{\nabla}^k\mathsf{Rol})_q(X,Y,Z_1,...,Z_k)A^T\in\mathfrak{so}(T_{\hat{x}}\hat{M})\ if\ n\geq\hat{n},$$

and therefore, $\nu(\mathsf{Rol}_q(X,Y))$, $(\overline{\nabla}^k\mathsf{Rol})_q(X,Y,Z_1,...,Z_k)$ are well defined as elements of $V|_q(\pi_Q)$.

Remark 4.1.5. With this notation, Eq. (4.3) can be written as

$$[\mathscr{L}_R(X),\mathscr{L}_R(Y)]|_q = \mathscr{L}_R([X,Y])|_q + \nu(\mathsf{Rol}_q(X,Y))|_q. \tag{4.4}$$

Proposition 4.1.6. *Let $X,\ Y,\ Z\in\mathrm{VF}(M)$. Then, for $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$, one has*

$$\begin{aligned}[\mathscr{L}_R(Z),\nu(\mathsf{Rol}(X,Y))]|_q =& -\mathscr{L}_{NS}(\mathsf{Rol}(X,Y)Z)|_q + \nu\big((\overline{\nabla}^1\mathsf{Rol})_q(X,Y,Z)\big)|_q \\ & + \nu\big(\mathsf{Rol}_q(\nabla_Z X,Y)\big)|_q + \nu\big(\mathsf{Rol}_q(X,\nabla_Z Y)\big)|_q.\end{aligned}$$

We recall the following notation we define

$$[A,B]_{\mathfrak{so}} := A\circ B - B\circ A\in\mathfrak{so}(T_xM).$$

Proposition 4.1.7. *Let $q = (x, \hat{x}; A) \in Q$ and X, Y, Z, $W \in \mathrm{VF}(M)$. We have*

$$\begin{aligned}
&\big[\nu(\mathsf{Rol}(X,Y)), \nu(\mathsf{Rol}(Z,W))\big]|_q \\
=&\nu\Big(A[R(X,Y),R(Z,W)]_{\mathfrak{so}} - [\hat{R}(AX,AY),\hat{R}(AZ,AW)]_{\mathfrak{so}}A - \hat{R}(\mathsf{Rol}_q(X,Y)Z,AW)(A) \\
&- \hat{R}(AZ,\mathsf{Rol}_q(X,Y)W)(A) + \hat{R}(AX,\mathsf{Rol}_q(Z,W)Y)(A) + \hat{R}(\mathsf{Rol}_q(Z,W)X,AY)\Big)|_q.
\end{aligned}$$

Proof. Cf. the proof of Proposition 5.18 and Corollary 5.19 of Section 5 in [10].

Proposition 4.1.8. *Consider the following smooth right and left actions of $Iso(M,g)$ and $Iso(\hat{M},\hat{g})$ on Q given by*

$$q_0 \cdot F := (F^{-1}(x_0), \hat{x}_0; A_0 \circ F_*|_{F^{-1}(x_0)}), \quad \hat{F} \cdot q_0 := (x_0, \hat{F}(\hat{x}_0); \hat{F}_*|_{\hat{x}_0} \circ A_0),$$

where $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, $F \in Iso(M,g)$ and $\hat{F} \in Iso(\hat{M},\hat{g})$. We also set

$$\hat{F} \cdot q_0 \cdot F := (\hat{F} \cdot q_0) \cdot F = \hat{F} \cdot (q_0 \cdot F).$$

Then for any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, absolutely continuous $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$, $F \in Iso(M,g)$ and $\hat{F} \in Iso(\hat{M},\hat{g})$, we have

$$\hat{F} \cdot q_{\mathscr{D}_R}(\gamma, q_0)(t) \cdot F = q_{\mathscr{D}_R}(F^{-1} \circ \gamma, \hat{F} \cdot q_0 \cdot F)(t), \tag{4.5}$$

for all $t \in [0,1]$. In particular, $\hat{F} \cdot \mathcal{O}_{\mathscr{D}_R}(q_0) \cdot F = \mathcal{O}_{\mathscr{D}_R}(\hat{F} \cdot q_0 \cdot F)$.

Proof. Cf. the proof of Proposition 5.5 of Section 5 in [10]. □

Remark 4.1.9. When $n \leq \hat{n}$, the right action of $Iso(M,g)$ on Q is free. Indeed, given F, $F' \in Iso(M,g)$, the existence of an $q = (x, \hat{x}; A) \in Q$ such that $q \cdot F = q \cdot F'$ implies that $F^{-1}(x) = F'^{-1}(x) := y$ and $A \circ F_*|_y = A \circ F'_*|_y$. Since $A^T A = \mathrm{id}$, we obtain $F_*|_y = F'_*|_y$, which implies, because M is connected, that $F = F'$ (see [32], page 43). The same argument proves the freeness of the left $Iso(\hat{M},\hat{g})$-action when $n \geq \hat{n}$.

4.1.2 Elementary Constructions when $\mid n - \hat{n} \mid = 1$

Proposition 4.1.10. *Let (M,g) and $(\hat{M},\hat{g})$ be Riemannian manifolds of dimensions n and $\hat{n} = n-1$ respectively, with $n \geq 2$. We use $(\hat{M}^{(1)}, \hat{g}^{(1)})$ to denote the Riemannian product $(\mathbb{R} \times \hat{M}, dr^2 \oplus \hat{g})$, where dr^2 denotes the canonical Riemannian metric on $\mathbb{R}$.*

Set $Q^{(1)} := Q(M, \hat{M}^{(1)})$ and let $\mathscr{L}_R^{(1)}$, $\mathscr{D}_R^{(1)}$ to be the rolling lift and the rolling distribution on $Q^{(1)}$. We define, for every $a \in \mathbb{R}$,

$$\iota_a : Q \to Q^{(1)}; \quad \iota_a(x, \hat{x}; A) = (x, (a, \hat{x}); A^{(1)}),$$

where $A^{(1)} : T_x M \to T_{(a,\hat{x})}(\mathbb{R} \times \hat{M})$ is defined as follows: $A^{(1)} \in Q^{(1)}$,

$$A^{(1)}|_{(\ker A)^\perp} = (0, A|_{(\ker A)^\perp}), \quad A^{(1)}(\ker A) = \mathbb{R}\partial_r|_{(a,\hat{x})} \times \{0\},$$

where ∂_r is the canonical vector field on $\mathbb{R}$ in the positive direction, also seen as a vector field on $\hat{M}^{(1)}$ in the usual way.
Then for every $a \in \mathbb{R}$, the map ι_a is an embedding and for every $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, $a_0 \in \mathbb{R}$ and $X \in T_x M$, one has

$$\mathscr{L}_R(X)|_{q_0} = \Pi_* \mathscr{L}_R^{(1)}(X)|_{\iota_{a_0}(q_0)},$$
$$\mathcal{O}_{\mathscr{D}_R}(q_0) = \Pi(\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))),$$

where

$$\begin{array}{cccc} \Pi : & Q^{(1)} & \to & Q; \\ & (x,(a,\hat{x});A^{(1)}) & \mapsto & (x,\hat{x};(pr_2)_* \circ A^{(1)}), \end{array}$$

is a surjective submersion and $pr_2 : \mathbb{R} \times \hat{M} \to \hat{M}$ is the projection onto the second factor.

Proof. Let γ be a path in M starting at x_0 and $q(t) = (\gamma(t), \hat{\gamma}(t); A(t)) := q_{\mathscr{D}_R}(\gamma, q_0)(t)$. We define a path $q^{(1)}(t) = (\gamma(t), \hat{\gamma}^{(1)}(t); A^{(1)}(t))$ on $Q^{(1)}$ as follows:

$$\hat{\gamma}^{(1)}(t) := \big(a_0 + \int_0^t \iota_{a_0}(A_0) p^T(A_0) P_s^0(\gamma)\dot{\gamma}(s) ds, \hat{\gamma}(t)\big), \quad A^{(1)} := P_0^t(\hat{\gamma}^{(1)}) \circ \iota_{a_0}(A_0) \circ P_t^0(\gamma),$$

where, for every $q = (x, \hat{x}; A) \in Q$, we define the g-orthogonal projections as

$$p^{\perp}(A) : T_x M \to (\ker A)^{\perp}, \quad p^T(A) : T_x M \to \ker A.$$

We will show that $q^{(1)}$ is the rolling curve on $Q^{(1)}$ starting from $\iota_{a_0}(q_0)$. Indeed, clearly $q^{(1)}(0) = (\gamma(0), (a_0, \hat{\gamma}(0)); \iota_{a_0}(A_0)) = \iota_{a_0}(q_0)$ and $A^{(1)}(t) \in Q^{(1)}$ for every time t and $\iota_{a_0}(A_0) \in Q^{(1)}$. We also have

$$\dot{\hat{\gamma}}^{(1)}(t) = (b(t)\partial_r|_{\hat{\gamma}^{(1)}(t)}, \dot{\hat{\gamma}}(t)),$$

where $b(t)$ is defined by $\iota_{a_0}(A_0) p^T(A_0) P_t^0(\gamma)\dot{\gamma}(t) := (b(t)\partial_r|_{(a_0,\hat{x}_0)}, 0)$. On the other hand,

$$\begin{aligned} A^{(1)}(t)\dot{\gamma}(t) &= P_0^t(\hat{\gamma}^{(1)})\iota_{a_0}(A_0)P_t^0(\gamma)\dot{\gamma}(t) \\ &= P_0^t(\hat{\gamma}^{(1)})\iota_{a_0}(A_0)(p^T(A_0) + p^{\perp}(A_0))P_t^0(\gamma)\dot{\gamma}(t). \end{aligned}$$

Since $\hat{M}^{(1)}$ is a Riemannian product, then, for every $\hat{X} \in T_{\hat{x}_0}\hat{M} \subset T_{(a_0,\hat{x}_0)}(\mathbb{R} \times \hat{M})$, we have

$$P_0^t(\hat{\gamma}^{(1)})(0, \hat{X}) = (0, P_0^t(\hat{\gamma})\hat{X}), \quad P_0^t(\hat{\gamma}^{(1)})(\partial_r|_{(a_0,\hat{x}_0)}, 0) = (\partial_r|_{\hat{\gamma}^{(1)}(t)}, 0).$$

However $\iota_{a_0}(A_0)p^{\perp}(A_0)X = (0, A_0 p^{\perp}(A_0)X) = (0, A_0 X)$ for every $X \in T_{x_0}M$, we get that

$$P_0^t(\hat{\gamma}^{(1)})\iota_{a_0}(A_0)p^{\perp}(A_0)P_t^0(\gamma)\dot{\gamma}(t) = P_0^t(\hat{\gamma}^{(1)})(0, A_0 P_t^0(\gamma)\dot{\gamma}(t)) = (0, P_0^t(\hat{\gamma})A_0 P_t^0(\gamma)\dot{\gamma}(t)),$$

and $P_0^t(\hat{\gamma}^{(1)})\iota_{a_0}(A_0)p^T(A_0)P_t^0(\gamma)\dot{\gamma}(t) = P_0^t(\hat{\gamma}^{(1)})(b(t)\partial_r|_{(a_0,\hat{x}_0)}, 0) = (b(t)\partial_r|_{\hat{\gamma}^{(1)}(t)}, 0)$. Therefore,

$$\begin{aligned} A^{(1)}(t)\dot{\gamma}(t) &= P_0^t(\hat{\gamma}^{(1)})\iota_{a_0}(A_0)(p^T(A_0) + p^{\perp}(A_0))P_t^0(\gamma)\dot{\gamma}(t) \\ &= (b(t)\partial_r|_{\hat{\gamma}^{(1)}(t)}, 0) + (0, P_0^t(\hat{\gamma})A_0 P_t^0(\gamma)\dot{\gamma}(t)) \\ &= (b(t)\partial_r|_{\hat{\gamma}^{(1)}(t)}, A(t)\dot{\gamma}(t)) \\ &= (b(t)\partial_r|_{\hat{\gamma}^{(1)}(t)}, \dot{\hat{\gamma}}(t)) = \dot{\hat{\gamma}}^{(1)}(t). \end{aligned}$$

This and the definition of $A^{(1)}(t)$ show that $q^{(1)}(t) = q_{\mathscr{D}_R^{(1)}}(\gamma, \iota_{a_0}(q_0))(t)$ for all t. Furthermore, since $A^{(1)}(t)\dot{\gamma}(t) = P_0^t(\hat{\gamma}^{(1)})\iota_{a_0}(A_0)(p^T(A_0) + p^{\perp}(A_0))P_t^0(\gamma)\dot{\gamma}(t)$ and by the basic properties of parallel transport, it follows that

$$\begin{aligned}\Pi(q_{\mathscr{D}_R^{(1)}}(\gamma, \iota_{a_0}(q_0))(t)) = \Pi(\gamma(t), \hat{\gamma}^{(1)}; A^{(1)}) &= (\gamma(t), \hat{\gamma}(t); (pr_2)_* \circ A^{(1)}) \\ &= (\gamma(t), \hat{\gamma}(t); (pr_2)_*(P_0^t(\hat{\gamma}^{(1)}) \circ \iota_{a_0}(A_0) \circ P_t^0(\gamma))) \\ &= (\gamma(t), \hat{\gamma}(t); P_0^t(\hat{\gamma})A_0P_t^0(\gamma)) = q_{\mathscr{D}_R}(\gamma, q_0)(t).\end{aligned}$$

Hence $\mathcal{O}_{\mathscr{D}_R}(q_0) \subset \Pi(\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0)))$ as well as

$$\Pi_*(\mathscr{L}_R^{(1)}(\dot{\gamma}(0))|_{\iota_{a_0}(q_0)}) = \Pi_*(\dot{q}_{\mathscr{D}_R^{(1)}}(\gamma, \iota_{a_0}(q_0))(0)) = \dot{q}_{\mathscr{D}_R}(\gamma, q_0)(0) = \mathscr{L}_R(\dot{\gamma}(0))|_{q_0}.$$

Finally, if $q^{(1)} = (x, (a, \hat{x}); A^{(1)}) \in \mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))$, take a path γ in M starting from x_0 such that $q^{(1)} = q_{\mathscr{D}_R^{(1)}}(\gamma, \iota_{a_0}(q_0))(1)$. By what was done above, it follows that $\Pi(q_{\mathscr{D}_R^{(1)}}(\gamma, \iota_{a_0}(q_0))(t)) = q_{\mathscr{D}_R}(\gamma, q_0)(t)$ and thus, evaluating this at $t = 1$ gives $\Pi(q^{(1)}) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, whence $\Pi(\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))) \subset \mathcal{O}_{\mathscr{D}_R}(q_0)$. The claim that ι_a is an embedding for every $a \in \mathbb{R}$ and Π is a surjective submersion are obvious from the fact $\Pi \circ \iota_a = \mathrm{id}_Q$. □

Corollary 4.1.11. *With the same notations of the previous proposition, if the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is not open in Q for some $q_0 \in Q$, then $\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))$ is not open in $Q^{(1)}$.*

Proof. Suppose that $\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))$ is open in $Q^{(1)}$. Since $\Pi : Q^{(1)} \to Q$ is a smooth submersion then it is an open map and hence its image $\Pi(\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))) = \mathcal{O}_{\mathscr{D}_R}(q_0)$ is open. □

With the assumption and the notations of Proposition 4.1.10, we have the following remark.

Remark 4.1.12. Keeping the same notations as before, recall that $Q = Q(M, \hat{M})$ is connected and thus, as a consequence of Corollary 4.1.11, if the system associated to the rolling of M and $\hat{M}^{(1)}$ is controllable then the system associated to the rolling of M and $\hat{M}$ is also controllable.

Proposition 4.1.13. *Let (M, g) and $(\hat{M}, \hat{g})$ be Riemannian manifolds of dimensions $n = \hat{n} - 1$ and $\hat{n}$, with $\hat{n} \geq 2$ respectively. Let $(M^{(1)}, g^{(1)})$ be the Riemannian product $(\mathbb{R} \times M, dr^2 \oplus g)$, with the obvious orientation. Write $Q^{(1)} = Q(M^{(1)}, \hat{M})$ and let $\mathscr{L}_R^{(1)}$, $\mathscr{D}_R^{(1)}$ be the rolling lift and the rolling distribution on $Q^{(1)}$. We define for every $a \in \mathbb{R}$,*

$$\iota_a : Q \to Q^{(1)}; \quad \iota_a(x, \hat{x}; A) = ((a, x), \hat{x}; A^{(1)}),$$

where $A^{(1)} : T_{(a,x)}(\mathbb{R} \times M) \to T_{\hat{x}}\hat{M}$ is defined as follows: $A^{(1)} \in Q^{(1)}$,

$$A^{(1)}|_{T_xM} = A, \quad A^{(1)}\partial_r|_{(a,x)} \in (\mathrm{im}A)^{\perp}.$$

Then for every $a \in \mathbb{R}$, the map ι_a is an embedding and for every $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, $a_0 \in \mathbb{R}$ and $X \in T_x M \subset T_{(a,x)}(\mathbb{R} \times M)$, one has

$$(\iota_{a_0})_* \mathscr{L}_R(X)|_{q_0} = \mathscr{L}_R^{(1)}(X)|_{\iota_{a_0}(q_0)}.$$

Moreover, if one defines

$$\begin{array}{rccl} \Pi : & Q^{(1)} & \to & Q; \\ & ((a,x), \hat{x}; A^{(1)}) & \mapsto & (x, \hat{x}; A^{(1)} \circ (i_a)_*), \end{array}$$

where $i_a : M \to \mathbb{R} \times M$; $x \mapsto (a, x)$ and if Δ_R is the subdistribution of $\mathscr{D}_R^{(1)}$ defined by

$$\Delta_R|_{q^{(1)}} = (\iota_a)_* \mathscr{D}_R|_{\Pi(q^{(1)})}, \quad \forall q^{(1)} = ((a,x), \hat{x}; A^{(1)}) \in Q^{(1)},$$

then $\iota_{a_0}(\mathcal{O}_{\mathscr{D}_R}(q_0)) = \mathcal{O}_{\Delta_R}(\iota_{a_0}(q_0)) \subset \mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))$.

Proof. The facts that ι_a is an embedding and Π is submersion simply follow from the fact $\Pi \circ \iota_a = \mathrm{id}_Q$. Let now γ be a path in M starting from x_0 and $q(t) = (\gamma(t), \hat{\gamma}(t); A(t)) = q_{\mathscr{D}_R}(\gamma, q_0)(t)$. We define a path $q^{(1)}(t) = (\gamma^{(1)}(t), \hat{\gamma}(t); A^{(1)}(t))$ on $Q^{(1)}$ by

$$\gamma^{(1)}(t) := (a_0, \gamma(t)), \quad A^{(1)} := P_0^t(\hat{\gamma}) \circ \iota_{a_0}(A_0) \circ P_t^0(\gamma^{(1)}),$$

We will show that $q^{(1)}$ is the rolling curve on $Q^{(1)}$ starting from $\iota_{a_0}(q_0)$. Indeed, clearly $q^{(1)}(0) = ((a_0, \gamma(0)), \hat{\gamma}(0); \iota_{a_0}(A_0)) = \iota_{a_0}(q_0)$ and for $\iota_{a_0}(A_0) \in Q^{(1)}$ we have $A^{(1)}(t) \in Q^{(1)}$ for all t. We also have $\dot{\gamma}^{(1)}(t) = (0, \dot{\gamma}(t))$. On the other hand,

$$\begin{aligned} A^{(1)}(t)\dot{\gamma}^{(1)}(t) &= P_0^t(\hat{\gamma})\iota_{a_0}(A_0)P_t^0(\gamma^{(1)})\dot{\gamma}^{(1)}(t) \\ &= P_0^t(\hat{\gamma})\iota_{a_0}(A_0)P_t^0(\gamma^{(1)})(0, \dot{\gamma}(t)). \end{aligned}$$

Since $M^{(1)}$ is a Riemannian product, then $P_t^0(\gamma^{(1)})(0, X) = (0, P_t^0(\gamma)X)$ for every $X \in T_{x_0}M \subset T_{(a_0,x_0)}(\mathbb{R} \times M)$. Therefore,

$$\begin{aligned} A^{(1)}(t)\dot{\gamma}^{(1)}(t) &= P_0^t(\hat{\gamma})\iota_{a_0}(A_0)(0, P_t^0(\gamma)\dot{\gamma}(t)) \\ &= P_0^t(\hat{\gamma})A_0 P_t^0(\gamma)\dot{\gamma}(t) \\ &= A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t). \end{aligned}$$

This proves that $q^{(1)}(t) = q_{\mathscr{D}_R^{(1)}}(\gamma^{(1)}, \iota_{a_0}(q_0))(t)$ for all t. Furthermore, notice that $\pi_{Q^{(1)}}(\iota_{a_0}(q(t))) = ((a_0, \gamma(t)), \hat{\gamma}(t)) = (\gamma^{(1)}(t), \hat{\gamma}(t)) = \pi_{Q^{(1)}}(q^{(1)}(t))$ and $A^{(1)}(t)(0, X) = A(t)X = \iota_{a_0}(A(t))X$ for every $X \in T_x M \subset T_{(a_0,x)}(\mathbb{R} \times M)$. However $A^{(1)}(t)T_{\gamma(t)}M \perp A^{(1)}(t)\partial_r|_{\gamma^{(1)}(t)}$ and $(\iota_{a_0} \circ A(t))T_x M \perp (\iota_{a_0} \circ A(t))\partial_r|_{\gamma^{(1)}(t)}$, we must have, by orientation, $A^{(1)}(t)\partial_r|_{\gamma^{(1)}(t)} = (\iota_{a_0} \circ A(t))\partial_r|_{\gamma^{(1)}(t)}$. This proves that $\iota_{a_0}(q(t)) = q^{(1)}(t)$ and hence

$$(\iota_{a_0})_* \mathscr{L}_R(\dot{\gamma}(0))|_{q_0} = (\iota_{a_0})_* \dot{q}(0) = \dot{q}^{(1)}(0) = \mathscr{L}_R^{(1)}(\dot{\gamma}^{(1)}(0))|_{\iota_{a_0}(q_0)} = \mathscr{L}_R^{(1)}((0, \dot{\gamma}(0)))|_{\iota_{a_0}(q_0)}.$$

So, $(\iota_{a_0})_* \mathscr{L}_R(X)|_{q_0} = \mathscr{L}_R^{(1)}(X)|_{\iota_{a_0}(q_0)}$ for every $X \in T_{x_0}M \subset T_{(a_0,x_0)}(\mathbb{R} \times M)$, then,

$$\iota_{a_0}(\mathcal{O}_{\mathscr{D}_R}(q_0)) \subset \mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0)).$$

Finally, recall that $\Pi \circ \iota_a = \mathrm{id}_Q$, then, for every $q \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, we have

$$\Delta_R|_{\iota_{a_0}(q)} = (\iota_{a_0})_*\mathscr{D}_R|_q \subset T_{\iota_{a_0}(q_0)}(\iota_{a_0}(\mathcal{O}_{\mathscr{D}_R}(q_0))).$$

Thus, one can write $\Delta_R|_{\iota_{a_0}(\mathcal{O}_{\mathscr{D}_R}(q_0))} = (\iota_{a_0})_*\mathscr{D}_R|_{\mathcal{O}_{\mathscr{D}_R}(q_0)}$. Then, $\iota_{a_0}(\mathcal{O}_{\mathscr{D}_R}(q_0)) \subseteq \mathcal{O}_{\Delta_R}(\iota_{a_0}(q_0))$. Since $\iota_{a_0}|_{\mathcal{O}_{\mathscr{D}_R}(q_0)}$ is an immersion, we get the equality $\iota_{a_0}(\mathcal{O}_{\mathscr{D}_R}(q_0)) = \mathcal{O}_{\Delta_R}(\iota_{a_0}(q_0))$. $\square$

Corollary 4.1.14. *With the assumptions of the previous proposition, if the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is open in Q for some $q_0 \in Q$, then the codimension of $\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))$ in $Q^{(1)}$ is at most 1.*

Proof. The relation between the dimension of Q and that of $Q^{(1)}$ is

$$\dim Q = 2\hat{n} - 1 + \frac{\hat{n}(\hat{n}-1)}{2} = \dim Q^{(1)} - 1.$$

On the other hand, if $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is open in Q then one has $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = \dim Q$. Thus,

$$\dim \mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0)) \geq \dim \mathcal{O}_{\Delta_R}(\iota_{a_0}(q_0)) = \dim \iota_{a_0}(\mathcal{O}_{\mathscr{D}_R}(q_0)) = \dim Q = \dim Q^{(1)} - 1.$$

$\square$

Theorem 4.1.15. *Let M and $\hat{M}$ be Riemannian manifolds of dimension $n = 3$ and $\hat{n} = 2$ respectively. If, for some $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is not open in Q, then there exists an open dense subset O of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ such that for every $q_1 = (x_1, \hat{x}_1; A_1) \in O$ there is an open neighbourhood U of x_1 for which it holds that $(U, g\mid_U)$ is isometric to some warped product $(I \times N, h_f)$, where $I \subset \mathbb{R}$ is an open interval and the warping function f satisfying $f'' = 0$.*

Proof. We will proceed by using Proposition 4.1.10. Let $(M^{(1)}, g^{(1)})$ be the Riemannian product $(\mathbb{R} \times \hat{M}, dr^2 \oplus \hat{g})$ and let $a_0 \in \mathbb{R}$. Since the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is not open in Q, it follows from Corollary 4.1.11 that $\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))$ is not open in $Q^{(1)}$. Theorem 7.1 of Section 7 in [10] provides an open subset $O^{(1)}$ of $\mathcal{O}_{\mathscr{D}_R^{(1)}}(\iota_{a_0}(q_0))$ such that one of $(a) - (c)$ of this theorem holds. So, $O := \Pi(O^{(1)})$ is a dense open of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ and let$q_1 = (x_1, \hat{x}_1; A_1) \in O$, then choose $q_1^{(1)} \in O^{(1)}$ such that $\Pi(q_1^{(1)}) = q_1$, whence $q_1^{(1)} = \iota_{a_1}(q_1)$ for some $a_1 \in \mathbb{R}$. Moreover, if U and $\hat{U}^{(1)}$ are the neighborhoods of x_1 and $(a_1, \hat{x}_1)$, respectively, as in Theorem 7.1 mentioned before, then we can choose $\hat{U}^{(1)}$ to be of the form $I \times \hat{U}$ for some open interval $I \subset \mathbb{R}$ and open neighborhood $\hat{U} \subset \hat{M}$ of $\hat{x}_1$. We consider the possible subcases.
If (a) holds, then $(U, g\mid_U)$ is (locally) isometric to the Riemannian product $I \times \hat{U}$, hence we have $f = 1$. If (b) holds, then $(U, g\mid_U)$ and $(\hat{U}^{(1)}, g^{(1)}\mid_{\hat{U}^{(1)}})$ are both of class $\mathcal{M}_\beta$ for some $\beta > 0$, but $(\hat{U}^{(1)}, g^{(1)}\mid_{\hat{U}^{(1)}})$ is as a Riemannian product, so it cannot be of such class $\mathcal{M}_\beta$, thus this case cannot occur. If (c) holds, let $F : (I \times N, h_f) \to U$ and $\hat{F} : (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}}) \to \hat{U}$ be the isomorphisms, it means that $(\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$ is isomorphic to a Riemannian product which implies $\hat{f}$ must satisfy $\hat{f}'' = 0$ thus also $f'' = 0$. $\square$

4.2 Controllability Results

4.2.1 The Rolling Problem $(\Sigma)_{NS}$

We start by the following remark about the non-compatibility of the (NS) system in the space $T^*M \otimes T\hat{M}$.

Remark 4.2.1. In the rolling system (NS), the distribution $\mathscr{D}_{NS}$ in never completely controllable in $T^*M \otimes T\hat{M}$. In order to prove this claim, define for any $k = 0, 1, ..., \min\{n, \hat{n}\}$,

$$r_k(M, \hat{M}) := \{(x, \hat{x}; A) \in T^*M \otimes T\hat{M} \mid A \text{ had rank } k\}.$$

Let $q = (x, \hat{x}; A) \in r_k(M, \hat{M})$, then $\mathcal{O}_{\mathscr{D}_{NS}}(q) \subset r_k(M, \hat{M})$. Indeed, for any $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathscr{D}_{NS}}(q)$, there exists an a.c. curve $q(t) := (\gamma(t), \hat{\gamma}(t); A(t))$ in $T^*M \otimes T\hat{M}$ joining $q(0) = q$ and $q(1) = q_1$ such that $A(t) = (P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma)$ (see Proposition 3.1.6). Since the parallel transport defines an invertible mapping and A has rank k, then $A(t)$ has also rank k. Thus rank $A_1 =$ rank $A(1) = k$, i.e. $q_1 \in r_k(M, \hat{M})$. On the other hand, $r_k(M, \hat{M})$ is obviously a submanifold of $T^*M \otimes T\hat{M}$. So, $\mathcal{O}_{\mathscr{D}_{NS}}(q)$ is a submanifold of $r_k(M, \hat{M})$ which cannot be equal to the whole manifold $T^*M \otimes T\hat{M}$.

Theorem 4.1.1 states that the controllability of $\mathscr{D}_{NS}$ is completely determined by the holonomy groups of M and $\hat{M}$. The next theorem highlights that fact at the Lie algebraic level.

Theorem 4.2.2. *Fix some orthonormal frames F, $\hat{F}$ of M, $\hat{M}$ at x and $\hat{x}$ respectively. Let $\mathfrak{h} := \mathfrak{h}|_F \subset \mathfrak{so}(n)$ and $\hat{\mathfrak{h}} := \hat{\mathfrak{h}}|_{\hat{F}} \subset \mathfrak{so}(\hat{n})$ be the holonomy Lie algebras of M and $\hat{M}$ with respect to these frames. Then the control system $(\Sigma)_{NS}$ is completely controllable if and only if for every $A \in \mathrm{SO}(n, \hat{n})$ (defined in (1.2)),*

$$\hat{\mathfrak{h}}A - A\mathfrak{h} = \begin{cases} \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid A^T B \in \mathfrak{so}(n)\}, & if\ n < \hat{n}, \\ \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid BA^T \in \mathfrak{so}(\hat{n})\}, & if\ n > \hat{n}. \end{cases} \tag{4.6}$$

Proof. By connectedness of Q, we get that $\mathscr{D}_{NS}$ is controllable if and only if every $\mathcal{O}_{\mathscr{D}_{NS}}(q)$, $q = (x, \hat{x}; A) \in Q$, is open in Q. Clearly, an orbit $\mathcal{O}_{\mathscr{D}_{NS}}(q_0) = Q$, $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, is an open subset of Q if and only if $T_q\mathcal{O}_{\mathscr{D}_{NS}}(q_0) = T_qQ$ for some (and hence every) $q \in \mathcal{O}_{\mathscr{D}_{NS}}(q_0)$. Thus the decomposition given by Remark 3.1.9 implies that an orbit $\mathcal{O}_{\mathscr{D}_{NS}}(q_0)$ is open in Q if and only if $V|_q(\pi_Q) \subset T_q\mathcal{O}_{\mathscr{D}_{NS}}(q_0)$ for some $q \in \mathcal{O}_{\mathscr{D}_{NS}}(q_0)$.
Fix $(x_0, \hat{x}_0) \in M \times \hat{M}$. Theorem 4.1.1 implies that every $\mathscr{D}_{NS}$-orbit intersects every π_Q-fiber. Hence $\mathscr{D}_{NS}$ is controllable if and only if $V|_q(\pi_Q) \subset T_q\mathcal{O}_{\mathscr{D}_{NS}}(q_0)$ for every $q = (x_0, \hat{x}_0; A) \in Q|_{(x_0,\hat{x}_0)}$. By (4.2), this condition is equivalent to the condition that, for every $q = (x_0, \hat{x}_0; A) \in Q|_{(x_0,\hat{x}_0)}$,

$$\nu(\hat{\mathfrak{h}}|_{\hat{x}_0} \circ A - A \circ \mathfrak{h}|_{x_0})|_{q_0} = V|_q(\pi_Q).$$

By Proposition 3.1.4, one can deduces that, for every $q \in Q$,

$$V|_q(\pi_Q) = \begin{cases} \nu(\{B \in T^*_{x_0}M \otimes T_{\hat{x}_0}\hat{M} \mid A^{\overline{T}}B \in \mathfrak{so}(T_{x_0}M)\})|_q, & if\ n \leq \hat{n}, \\ \nu(\{B \in T^*_{x_0}M \otimes T_{\hat{x}_0}\hat{M} \mid BA^{\overline{T}} \in \mathfrak{so}(T_{\hat{x}_0}\hat{M})\})|_q, & if\ n \geq \hat{n}. \end{cases}$$

Thus, we conclude that $\mathscr{D}_{NS}$ is controllable if and only if, for all $q = (x_0, \hat{x}_0; A) \in Q|_{(x_0,\hat{x}_0)}$

$$\hat{\mathfrak{h}}|_{\hat{x}_0} \circ A - A \circ \mathfrak{h}|_{x_0} = \begin{cases} \{B \in T^*_{x_0}M \otimes T_{\hat{x}_0}\hat{M} \mid A^{\overline{T}}B \in \mathfrak{so}(T_{x_0}M)\}, & if\ n \leq \hat{n}, \\ \{B \in T^*_{x_0}M \otimes T_{\hat{x}_0}\hat{M} \mid BA^{\overline{T}} \in \mathfrak{so}(T_{\hat{x}_0}\hat{M})\}, & if\ n \geq \hat{n}. \end{cases}$$

Choosing arbitrary orthonormal local frames F and $\hat{F}$ of M and $\hat{M}$ at x_0 and $\hat{x}_0$, respectively, we see that the above condition is equivalent to

$$\hat{\mathfrak{h}}|_{\hat{F}}\mathcal{M}_{F,\hat{F}}(A) - \mathcal{M}_{F,\hat{F}}(A)\mathfrak{h}|_F = \begin{cases} \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid \mathcal{M}_{F,\hat{F}}(A)^{\overline{T}}B \in \mathfrak{so}(n)\}, & if\ n \leq \hat{n}, \\ \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid B\mathcal{M}_{F,\hat{F}}(A)^{\overline{T}} \in \mathfrak{so}(\hat{n})\}, & if\ n \geq \hat{n}. \end{cases}$$

Since we have $\{\mathcal{M}_{F,\hat{F}}(A) \mid A \in Q|_{(x_0,\hat{x}_0)}\} = \mathrm{SO}(n,\hat{n})$, $T^*M \otimes T\hat{M} \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}}$ and $F, \hat{F}$ were arbitrary chosen, the claim follows. $\square$

Theorem 4.2.3. *Suppose that M, $\hat{M}$ are simply connected. Then $(\Sigma)_{NS}$ is completely controllable if and only if*

$$\hat{\mathfrak{h}}I_{n,\hat{n}} - I_{n,\hat{n}}\mathfrak{h} = \begin{cases} \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid I^T_{n,\hat{n}}B \in \mathfrak{so}(n)\}, & if\ n \leq \hat{n}, \\ \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid BI^T_{n,\hat{n}} \in \mathfrak{so}(\hat{n})\}, & if\ n \geq \hat{n}. \end{cases} \tag{4.7}$$

Proof. Notice that $I_{n,\hat{n}} \in \mathrm{SO}(n,\hat{n})$, then the previous theorem give the necessary condition.
Conversely, suppose that the condition (4.7) holds. This condition implies that for $(x_0, \hat{x}_0) \in M \times \hat{M}$, there is an $q_0 = (x_0, \hat{x}_0; A_0) \in Q|_{(x_0,\hat{x}_0)}$ such that

$$\hat{\mathfrak{h}}A_0 - A_0\mathfrak{h} = \begin{cases} \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid A_0^T B \in \mathfrak{so}(n)\}, & if\ n \leq \hat{n}, \\ \{B \in (\mathbb{R}^n)^* \otimes \mathbb{R}^{\hat{n}} \mid BA_0^T \in \mathfrak{so}(\hat{n})\}, & if\ n \geq \hat{n}. \end{cases}$$

By Proposition 3.1.4 and the equality (4.2), this means that $T_{q_0}\mathcal{O}_{\mathscr{D}_{NS}}(q_0) \cap V|_{q_0}(\pi_Q) = V|_{q_0}(\pi_Q)$ and hence $T_{q_0}\mathcal{O}_{\mathscr{D}_{NS}}(q_0) = T_{q_0}Q$ due to Remark 3.1.9. Thus $\mathcal{O}_{\mathscr{D}_{NS}}(q_0)$ is open in Q. By the connectedness of Q, we have that $\mathcal{O}_{\mathscr{D}_{NS}}(q_0) = Q$. Therefore, $(\Sigma)_{NS}$ is completely controllable. $\square$

Remark 4.2.4. The proofs of Theorems 4.2.2 and 4.2.3 are similar to that of Theorems 4.8 and 4.9 of Section 4 in [10].

4.2.2 The Rolling Problem $(\Sigma)_R$

From Proposition 3.1.16, we get the subsequent proposition and corollary whose proofs follow those of Proposition 5.20 and Corollary 5.21 of Section 5 in [10].

Proposition 4.2.5. *Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. Suppose that, for some $X \in \mathrm{VF}(M)$ and for a real sequence $(t_n)_{n=1}^{\infty}$ which verifies $t_n \neq 0$ for all n and $\lim_{n\to\infty} t_n = 0$, we have*

$$V|_{\Phi_{\mathscr{L}_R(X)}(t_n,q_0)}(\pi_Q) \subset T(\mathcal{O}_{\mathscr{D}_R}(q_0)), \quad \forall n. \tag{4.8}$$

Then $\mathscr{L}_{NS}(Y,\hat{Y})|_{q_0} \in T_{q_0}(\mathcal{O}_{\mathscr{D}_R}(q_0))$ for every Y g-orthogonal to $X|_{x_0}$ in $T_{x_0}M$ and every $\hat{Y}$ $\hat{g}$-orthogonal to $A_0X|_{x_0}$, $\in A_0(X|_{x_0})^{\perp}$ in $T_{\hat{x}_0}\hat{M}$. Hence the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ has codimension at most $|\hat{n}-n|+1$ inside Q.

Corollary 4.2.6. *Suppose there is a point $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and $\epsilon > 0$ such that for every $X \in \mathrm{VF}(M)$ with $\|X\|_g < \epsilon$ on M, one has*

$$V|_{\Phi_{\mathscr{L}_R(X)}(t,q_0)}(\pi_Q) \subset T\mathcal{O}_{\mathscr{D}_R}(q_0), \quad |t| < \epsilon.$$

Then the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is open in Q. As a consequence, $(\Sigma)_R$ is completely controllable if and only if

$$\forall q \in Q, \quad V|_q(\pi_Q) \subset T_q\mathcal{O}_{\mathscr{D}_R}(q). \tag{4.9}$$

Remark 4.2.7. We will use in the next corollary the fact that we have $\mathscr{D}_R|_q$ is involutive if and only if Rol_q is vanish for all $q = (x, \hat{x}; A) \in Q$, i.e. if and only if $\hat{R}(AX, AY)(AZ) = A(R(X,Y)Z)$, for all $X, Y, Z \in T_xM$. This is an immediate result from the equality (4.3) and the decomposition of Remark 3.1.9.

Corollary 4.2.8. *Assume that $n \leq \hat{n}$. Then the following two cases are equivalent,*

(i) *$\mathscr{D}_R$ is involutive,*

(ii) *(M,g) and $(\hat{M},\hat{g})$ have constant and equal curvature.*

Otherwise, i.e. if $n > \hat{n}$, then the following two cases are equivalent,

(a) *$\mathscr{D}_R$ is involutive,*

(b) *(M,g) and $(\hat{M},\hat{g})$ are both flat.*

Proof. The proof of $(i) \Leftrightarrow (ii)$ is similar to that of Corollary 5.23 of Section 5 in [10]. We next turn to the proof of $(a) \Rightarrow (b)$. Assume that $\widehat{\mathscr{D}_R}$ is involutive i.e., for every $\hat{q} = (\hat{x}, x; B) \in \hat{Q}$, $\quad \hat{X}, \hat{Y}, \hat{Z} \in T_{\hat{x}}\hat{M}$,

$$\widehat{Rol}_{\hat{q}}(\hat{X},\hat{Y})\hat{Z} = B(\hat{R}(\hat{X},\hat{Y})\hat{Z}) - R(B\hat{X}, B\hat{Y})(B\hat{Z}) = 0.$$

Thus, we have, for any $X, Y \in T_xM$,

$$\sigma_{(X,Y)} = g(R(X,Y)Y,X) = g(R(BB^TX, BB^TY)(BB^TY),X) = g(B(\hat{R}(B^TX, B^TY)(B^TY)),X).$$

Since $g(B(\hat{R}(B^TX, B^TY)(B^TY)), X) = \hat{g}(\hat{R}(B^TX, B^TY)(B^TY), B^TX)$, one deduces that $\sigma_{(X,Y)}$ is equal to $\hat{\sigma}_{(B^TX,B^TY)}$. Given any $x \in M$, $\hat{x} \in \hat{M}$, X, $Y \in T_xM$ and $\hat{X}$, $\hat{Y} \in T_{\hat{x}}\hat{M}$, choose some vectors $X_3, ..., X_n \in T_xM$ and $\hat{X}_3, ..., \hat{X}_{\hat{n}} \in T_{\hat{x}}\hat{M}$ such that $X, Y, X_3, ..., X_n$ and $\hat{X}, \hat{Y}, \hat{X}_3, ..., \hat{X}_{\hat{n}}$ are positively oriented orthonormal frames. We define

$$B\hat{X} = X, B\hat{Y} = Y, \quad B\hat{X}_i = 0; \quad i = 3, ..., n, \quad B\hat{X}_i = 0; \quad i = n+1, ..., \hat{n}.$$

Clearly, $\hat{q} = (\hat{x}, x; B) \in \hat{Q}$ and $\sigma_{(X,Y)} = \hat{\sigma}_{(\hat{X},\hat{Y})}$ for $B^TX = \hat{X}$, $B^TY = \hat{Y}$. Thus (M, g) and $(\hat{M}, \hat{g})$ have equal and constant curvature $k \in \mathbb{R}$. We need to show that $k = 0$. Choose any $(\hat{x}, x; B) \in \hat{Q}$, since $n < \hat{n}$, choose non-zero vectors $\hat{X} \in \ker B$ and $\hat{Y} \in (\ker B)^\perp$ and compute

$$0 = \widehat{Rol}(\hat{X}, \hat{Y})(B)\hat{X} = k(\hat{g}(\hat{Y}, \hat{X})B\hat{X} - \hat{g}(\hat{X}, \hat{X})B\hat{Y}) - R(B\hat{X}, B\hat{Y})(B\hat{X}) = -k\|\hat{X}\|_{\hat{g}}^2 B\hat{Y}.$$

However $\|\hat{X}\|_{\hat{g}} \neq 0$ and $B\hat{Y} \neq 0$, it follows that $k = 0$.

We now prove that $(b) \Rightarrow (a)$. In the case where (M, g) and $(\hat{M}, \hat{g})$ are flat, we have $R = 0$ and $\hat{R} = 0$ so that clearly $\widehat{Rol}(\hat{X}, \hat{Y})(B)\hat{Z} = B(\hat{R}(\hat{X}, \hat{Y})\hat{Z}) - R(B\hat{X}, B\hat{Y})(B\hat{Z}) = 0$ for all $(\hat{x}, x; B) \in \hat{Q}$ and $\hat{X}$, $\hat{Y}$, $\hat{Z} \in T_{\hat{x}}\hat{M}$. This proves that $\widehat{\mathscr{D}_R}$ is involutive. $\square$

We have another equivalence relation similar to Corollary 5.24 of Section 5 in [10].

Proposition 4.2.9. *Suppose that (M, g) and $(\hat{M}, \hat{g})$ are complete. The following cases are equivalent:*

(i) *There exists a $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is an integral manifold of $\mathscr{D}_R$.*

(ii) *There exists a $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that,*

$$\mathsf{Rol}_q(X, Y) = 0, \quad \forall q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0), \ X, Y \in T_xM.$$

(iii) *There is a complete Riemannian manifold (N, h), a Riemannian covering map $F : N \to M$ and a smooth map $G : N \to \hat{M}$ such that*

  (1) *If $n \leq \hat{n}$, G is a Riemannian immersion that maps h-geodesics to $\hat{g}$-geodesics.*

  (2) *If $n \geq \hat{n}$, G is a Riemannian submersion such that the co-kernel distribution $(\ker G_*)^\perp \subset TN$ is involutive and the fibers $G^{-1}(\hat{x})$, $\hat{x} \in \hat{M}$, are totally geodesic submanifolds of (N, h).*

Proof. We will first establish the equivalence $(i) \iff (ii)$ and to complete the proof, we proceed to show that $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$.

We prove $(i) \Rightarrow (ii)$. Notice that the restrictions of vector fields $\mathscr{L}_R(X)$, with $X \in \mathrm{VF}(M)$, to the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ are smooth vector fields of that orbit. Thus $[\mathscr{L}_R(X), \mathscr{L}_R(Y)]$ is tangent to this orbit for any X, $Y \in \mathrm{VF}(M)$ and hence (4.3)) implies the claim.

We next prove $(ii) \Rightarrow (i)$. From (4.3), it also follows that $\mathscr{D}_R|_{\mathcal{O}_{\mathscr{D}_R}(q_0)}$, the restriction of $\mathscr{D}_R$ to the manifold $\mathcal{O}_{\mathscr{D}_R}(q_0)$, is involutive. Since the maximal connected integral manifolds of an involutive distribution are exactly its orbits, we get that $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is an integral manifold of $\mathscr{D}_R$.

We now prove $(i) \Rightarrow (iii)$. Let $N := \mathcal{O}_{\mathscr{D}_R}(q_0)$ and $h := (\pi_{Q,M}|_N)^*(g)$ i.e. for $q = (x, \hat{x}; A) \in N$ and X, $Y \in T_xM$, define

$$h(\mathscr{L}_R(X)|_q, \mathscr{L}_R(Y)|_q) = g(X, Y).$$

If $F := \pi_{Q,M}|_N$ and $G := \pi_{Q,\hat{M}}|_N$, we immediately see that F is a local isometry (note that $\dim(N) = n$). The completeness of (N, h) follows from the completeness of M and $\hat{M}$ with Remark 3.1.13. Hence F is a surjective Riemannian covering. Moreover, if $\overline{\Gamma} : [0, 1] \to N$ is a h-geodesic, it is tangent to $\mathscr{D}_R$ and since it projects by F to a g-geodesic γ, it follows again by Remark 3.1.13 that $G \circ \overline{\Gamma} = \hat{\gamma}_{\mathscr{D}_R}(\gamma, \overline{\Gamma}(0))$ is a $\hat{g}$-geodesic. Therefore we have proven that G is a totally geodesic mapping $N \to \hat{M}$.

If $n \leq \hat{n}$, then for $q = (x, \hat{x}; A) \in N$, X, $Y \in T_xM$, one has

$$\hat{g}(G_*(\mathscr{L}_R(X)|_q), G_*(\mathscr{L}_R(Y)|_q)) = \hat{g}(AX, AY) = g(X, Y) = h_*(\mathscr{L}_R(X)|_q, \mathscr{L}_R(Y)|_q),$$

i.e. G is a Riemannian immersion. Item (1) is proved.

If $n \geq \hat{n}$, for $q = (x, \hat{x}; A) \in N$ and $X \in T_xM$ such that $\mathscr{L}_R(X)|_q \in (\ker G_*|_q)^\perp$ and $Z \in \ker A$, we have $G_*(\mathscr{L}_R(Z)|_q) = AZ = 0$ i.e. $\mathscr{L}_R(Z)|_q \in \ker(G_*|_q)$ from which $g(X, Z) = h(\mathscr{L}_R(X)|_q, \mathscr{L}_R(Z)|_q) = 0$ for all $Z \in \ker A$. This shows that $X \in (\ker A)^\perp$. Therefore, for all X, $Y \in T_xM$ such that $\mathscr{L}_R(X)|_q$, $\mathscr{L}_R(Y)|_q \in (\ker G_*|_q)^\perp$, we get $\hat{g}(G_*(\mathscr{L}_R(X)|_q), G_*(\mathscr{L}_R(Y)|_q)) = h(\mathscr{L}_R(X)|_q, \mathscr{L}_R(Y)|_q)$ as above. This proves that $G : N \to \hat{M}$ is a Riemannian submersion, which is also totally geodesic. It then follows from Theorem 3.3 in [36], that the fibers of G are totally geodesic submanifolds of N and that the co-kernel (i.e. horizontal) distribution $(\ker G_*)^\perp$ is involutive. Item (2), and hence the implication $(i) \Rightarrow (iii)$ has been proved.

We next prove $(iii) \Rightarrow (ii)$. Let $x_0 \in M$ and choose $z_0 \in N$ such that $F(z_0) = x_0$. Define $\hat{x}_0 = G(z_0) \in \hat{M}$ and $A_0 := G_*|_{z_0} \circ (F_*|_{z_0})^{-1} : T_{x_0}M \to T_{\hat{x}_0}\hat{M}$. The fact that $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ can be seen as follows: if $(iii) - (1)$ holds, we have

$$\hat{g}(A_0X, A_0Y) = \hat{g}(G_*|_{z_0}((F_*|_{z_0})^{-1}X), G_*|_{z_0}((F_*|_{z_0})^{-1}Y)) = h((F_*|_{z_0})^{-1}X, (F_*|_{z_0})^{-1}Y) = g(X, Y),$$

where we used that G is a Riemannian immersion. If $(iii) - (2)$ holds, take X, $Y \in (\ker A_0)^\perp$, clearly $(F_*|_{z_0})^{-1}X$, $(F_*|_{z_0})^{-1}Y \in (\ker G_*|_{z_0})^\perp$ and hence $\hat{g}(A_0X, A_0Y) = g(X, Y)$ because G is a Riemannian submersion.

Let $\gamma : [0, 1] \to M$ be a smooth curve with $\gamma(0) = x_0$. Since F is a smooth covering map, there is a unique smooth curve $\Gamma : [0, 1] \to N$ with $\gamma = F \circ \Gamma$ and $\Gamma(0) = z_0$. Define $\hat{\gamma} = G \circ \Gamma$ and $A(t) = G_*|_{\Gamma(t)} \circ (F_*|_{\Gamma(t)})^{-1}$, $t \in [0, 1]$. As before, it follows that $q(t) = (\gamma(t), \hat{\gamma}(t); A(t)) \in Q$ for all $t \in [0, 1]$ and

$$\dot{\hat{\gamma}}(t) = G_*|_{\Gamma(t)}\dot{\Gamma}(t) = A(t)\dot{\gamma}(t). \tag{4.10}$$

According to Theorem 3.3 in [36], the subcases (1) and (2) mean, respectively, that G is a totally geodesic map, which is moreover a Riemannian (1) immersion, (2) submersion. By Corollary 1.6 in [36], G is then affine map i.e. preserves parallel transport. But F, being a Riemannian covering map, also preserves parallel transport, i.e. is affine. It follows that $A(t) = G_*|_{\Gamma(t)} \circ (F_*|_{\Gamma(t)})^{-1}$ also preserves parallel transport, which combined with (4.10) means that $A(t)$ is the rolling curve along γ with $A(0) = A_0$.

Since the affinity of F (resp. G) simply means that $\nabla_{F_*\overline{X}}(F_*(\overline{Y})) = F_*(\nabla^h_{\overline{X}}\overline{Y})$ (resp. $\hat{\nabla}_{G_*\overline{X}}(G_*(\overline{Y})) = G_*(\nabla^h_{\overline{X}}\overline{Y})$) for all vector fields $\overline{X}, \overline{Y}$ on N, we easily see that

$$\begin{aligned} R(F_*\overline{X}, F_*\overline{Y})F_*\overline{Z} &= F_*(R^h(\overline{X}, \overline{Y})\overline{Z}) \\ \hat{R}(G_*\overline{X}, G_*\overline{Y})G_*\overline{Z} &= G_*(R^h(\overline{X}, \overline{Y})\overline{Z}), \end{aligned}$$

for all vector fields $\overline{X}, \overline{Y}, \overline{Z}$ on N. It thus follows that for all vector fields X, Y, Z on M,

$$\begin{aligned} A(t)(R(X,Y)Z) =& A(t)(R(F_*\overline{X}, F_*\overline{Y})F_*\overline{Z}) = A(t)(F_*(R^h(\overline{X}, \overline{Y})\overline{Z})) \\ =& G_*|_{\Gamma(t)}(R^h(\overline{X}, \overline{Y})\overline{Z})) = \hat{R}(G_*|_{\Gamma(t)}\overline{X}, G_*|_{\Gamma(t)}\overline{Y})G_*|_{\Gamma(t)}\overline{Z} \\ =& \hat{R}(A(t)X, A(t)Y)(A(t)Z), \end{aligned}$$

where $\overline{X}, \overline{Y}, \overline{Z}$ are any (local) F-lifts of X, Y, Z on N. This proves that

$$\mathsf{Rol}_{q(t)} = 0. \tag{4.11}$$

Thus we have shown that $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ is the unique rolling curve along γ starting at $q_0 = (x_0, \hat{x}_0; A_0)$ and defined on $[0, 1]$ and therefore curves of Q formed in this manner fill up the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Moreover, by Eq. (4.11) we have shown also that Rol vanishes on $\mathcal{O}_{\mathscr{D}_R}(q_0)$.

Remark 4.2.10. As pointed out in the course of the above proof, according to [36] the subcases (1)-(2) of (iii) in the previous proposition can be replaced by simply saying that G is a totally geodesic map which is a Riemannian (1) immersion, (2) submersion, respectively.

The next proposition is a sufficient condition of non-controllability for the rolling system $\Sigma_{(R)}$ when $n < \hat{n}$.

Proposition 4.2.11. *Let M, $\hat{M}$ be two Riemannian manifolds of dimensions n, $\hat{n}$ with $n < \hat{n}$. Assume that there exists a complete totally geodesic submanifold $\hat{N}$ of $\hat{M}$ of dimension m such that $n \le m < \hat{n}$. Then, the rolling system $\Sigma_{(R)}$ of $Q(M, \hat{M})$ is not completely controllable.*

Proof. Since $n \le m$, we can find $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that $\hat{x}_0 \in \hat{N}$ and $\mathrm{im}(A_0) \subset T_{\hat{x}_0}\hat{N}$. We proceed to prove that $\pi_{Q,\hat{M}}(\mathcal{O}_{\mathscr{D}_R}(q_0)) \subset \hat{N}$. To this end, we will first prove that for every geodesic curve γ on M starting at any point $q = (x, \hat{x}; A)$, with $x \in M$, $\hat{x} \in \hat{N}$ and $\mathrm{im}(A) \subset T_{\hat{x}}\hat{N}$, the resulting geodesic curve

$\hat{\gamma}_{\mathscr{D}_R} := \hat{\gamma}_{\mathscr{D}_R}(\gamma, q) = \pi_{Q,\hat{M}}(q_{\mathscr{D}_R}(\gamma, q))$ stays in $\hat{N}$ and that if $q_{\mathscr{D}_R}(\gamma, q) = (\gamma, \hat{\gamma}; A_{\mathscr{D}_R}(\gamma, q))$, then $\mathrm{im} A_{\mathscr{D}_R}(\gamma, q)(\cdot) \subset T_{\hat{\gamma}(\cdot)}\hat{N}$.

Once this proved, it is clearly obvious that the previous statement extends verbatim to the case where γ is any broken geodesic curve. By a standard density argument, we conclude that the above statement is again true for any absolutely continuous curve γ on M. We then prove the claim.

Let then consider a point $q = (x, \hat{x}; A)$, with $x \in M$, $\hat{x} \in \hat{N}$ and $\mathrm{im}(A) \subset T_{\hat{x}}\hat{N}$ and a geodesic curve $\gamma : [0, 1] \to M$ starting at $x \in M$. Then, $q_{\mathscr{D}_R}(\gamma, q)$ is a geodesic curve and so that $\hat{\gamma}_{\mathscr{D}_R}(\gamma, q)$ is a geodesic curve on $\hat{M}$ and for all $t \in [0, 1]$, we have,

$$\begin{aligned}\dot{\hat{\gamma}}_{\mathscr{D}_R}(t) =& \dot{\hat{\gamma}}_{\mathscr{D}_R}(\gamma, q)(t) = A_{\mathscr{D}_R}(\gamma, q)(t)\dot{\gamma}(t) = (P_0^t(\hat{\gamma}_{\mathscr{D}_R}) \circ A \circ P_t^0(\gamma))\dot{\gamma}(t)\\ =& P_0^t(\hat{\gamma}_{\mathscr{D}_R})(A\dot{\gamma}(0)).\end{aligned}$$

By assumption $\mathrm{im}(A) \subset T_{\hat{x}}\hat{N}$, and therefore $A\dot{\gamma}(0) \in T_{\hat{x}}\hat{N}$, which implies that $\dot{\hat{\gamma}}_{\mathscr{D}_R}(0) \in T_{\hat{x}}\hat{N}$. Since $\hat{N}$ is a complete totally geodesic submanifold of $\hat{M}$, we therefore have that the geodesic $\hat{\gamma}_{\mathscr{D}_R}(t)$ stays in N for all $t \in [0, 1]$.

Using the same reasoning, for a given $t \in [0, 1]$, if $X \in T_{\gamma(t)}M$, we have $A(P_t^0(\gamma)X) \in T_{\hat{x}}\hat{N}$, and hence, since $\hat{N}$ is totally geodesic, $A_{\mathscr{D}_R}(\gamma, q)(t)X \in T_{\hat{\gamma}(t)}\hat{N}$. This combined with the fact that $A_{\mathscr{D}_R}(\gamma, q)(t)$ preserves the inner product $\hat{g}$ of $\hat{M}$, and therefore that induced on $\hat{N}$, means that $q_{\mathscr{D}_R}(\gamma, q)(t) \in Q(M, \hat{N})$ for all $t \in [0, 1]$, which completes the proof.

Since Riemannian manifolds $(\hat{M}, \hat{g})$ of constant curvature contain complete totally geodesic submanifolds of any lower dimension, we get the following non-controllability result as consequence of the previous proposition.

Corollary 4.2.12. *Consider a Riemannian manifold (M, g) of dimension n and a Riemannian manifold $(\hat{M}, \hat{g})$ of constant curvature and of dimension $\hat{n} > n$. Then the rolling problem of (M, g) onto $(\hat{M}, \hat{g})$ without spinning nor slipping is not controllable.*

Chapter 5

Rolling of 2-dimensional against 3-dimensional Riemannian manifolds

In the latter chapter, we have provided in Theorem 4.1.15 all the necessary conditions for the non-controllability of rolling of 3-dimensional against 2-dimensional Riemannian manifolds. The situation is complicated when (M, g) and $(\hat{M}, \hat{g})$ are two oriented Riemannian manifolds of dimensions $n = 2$ and $\hat{n} = 3$ respectively. In the current chapter, we state some of their necessary conditions for the non-controllability issue. Before we start computing the tangent Lie brackets on $\mathcal{O}_{\mathscr{D}_R}(q_0)$, let us set some notations.

5.1 Preliminaries

Fix a $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and denote for the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ the following projections,

$$\begin{array}{rlrl}
\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0)} := \pi_Q \mid_{\mathcal{O}_{\mathscr{D}_R}(q_0)}: & \mathcal{O}_{\mathscr{D}_R}(q_0) \to & M \times \hat{M}, \\
\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0), M} := pr_1 \circ \pi_{\mathcal{O}_{\mathscr{D}_R}(q_0)}: & \mathcal{O}_{\mathscr{D}_R}(q_0) \to & M, \\
\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0), \hat{M}} := pr_2 \circ \pi_{\mathcal{O}_{\mathscr{D}_R}(q_0)}: & \mathcal{O}_{\mathscr{D}_R}(q_0) \to & \hat{M},
\end{array}$$

where $pr_1 : M \times \hat{M} \to M$ and $pr_2 : M \times \hat{M} \to \hat{M}$ are the projections onto the first and second factors respectively. The Hodge-duals of $(\hat{M}, \hat{g})$ will be denoted by $\star$.

Define for a $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ the local g-orthonormal frame X, $Y \in T_x M$. Set $\hat{Z}_A := \star(AX \wedge AY) \in T_{\hat{x}} M$, then we have AX, AY, $\hat{Z}_A$ is a local oriented orthonormal frame in $T_{\hat{x}} \hat{M}$.

Furthermore, we define the tensor $X \wedge Y$ as being the linear map $T_x M \to T_x M$ represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.1}$$

Moreover, if θ_X, θ_Y are the g-dual of X, Y respectively, then we remark that

$$\begin{aligned}
(\star \hat{Z}_A)A &= (AX \wedge AY)A = A(X \wedge Y),\\
(\star AY)A &= (\hat{Z}_A \wedge AX)A = -\theta_X \otimes \hat{Z}_A,\\
(\star AX)A &= (AY \wedge \hat{Z}_A)A = \theta_Y \otimes \hat{Z}_A.
\end{aligned}$$

We rewrite the Levi Civita connection on M by $\Gamma := \gamma^1 X + \gamma^2 Y$ where, by Koszul formula, we have $\gamma^1 := \Gamma^1_{(1,2)} = g(\nabla_X X, Y)$ and $\gamma^2 := \Gamma^2_{(1,2)} = g(\nabla_Y X, Y)$. Thus, the Gaussian curvature of M is given by

$$\begin{aligned}
K(x) := g(R(X,Y)Y,X) &= g((X(\Gamma^2_{(1,2)}) - Y(\Gamma^1_{(1,2)}) + ((\Gamma^1_{(1,2)})^2 + (\Gamma^2_{(1,2)})^2)X \wedge Y)Y, X)\\
&= Y(\Gamma^1_{(1,2)}) - X(\Gamma^2_{(1,2)}) - ((\Gamma^1_{(1,2)})^2 + (\Gamma^2_{(1,2)})^2)\\
&= g(\nabla_Y \Gamma, X) - g(\nabla_X \Gamma, Y).
\end{aligned} \tag{5.2}$$

The curvatures on $\hat{M}$ are as follows

$$\begin{aligned}
&\hat{\sigma}^1_A := \hat{g}(\hat{R}(AY, \hat{Z}_A)\hat{Z}_A, AY) = -\hat{g}(\hat{R}(\star AX), \star AX),\\
&\hat{\sigma}^2_A := \hat{g}(\hat{R}(AX, \hat{Z}_A)\hat{Z}_A, AX) = -\hat{g}(\hat{R}(\star AY), \star AY),\\
&\hat{\sigma}^3_A = \hat{\sigma}_A := \hat{g}(\hat{R}(AX, AY)AY, AX) = -\hat{g}(\hat{R}(\star \hat{Z}_A), \star \hat{Z}_A),\\
&\Pi_X(q) := \hat{g}(\hat{R}(\star \hat{Z}_A), \star AX),\\
&\Pi_Y(q) := \hat{g}(\hat{R}(\star \hat{Z}_A), \star AY),\\
&\Pi_Z(q) := \hat{g}(\hat{R}(\star AX), \star AY).
\end{aligned}$$

With respect to the previous notations, we have

$$\begin{aligned}
\mathsf{Rol}_q(X,Y) &= AR(X,Y) - \hat{R}(AX, AY)A\\
&= A((-K)(X \wedge Y)) - (\Pi_X \star AX + \Pi_Y \star AY + (-\hat{\sigma}_A) \star \hat{Z}_A)A\\
&= -(K - \hat{\sigma}_A)A(X \wedge Y) + \Pi_Y \theta_X \otimes \hat{Z}_A - \Pi_X \theta_Y \otimes \hat{Z}_A\\
&= \begin{pmatrix} 0 & -(K - \hat{\sigma}_A)\\ K - \hat{\sigma}_A & 0\\ \Pi_Y & -\Pi_X \end{pmatrix}.
\end{aligned}$$

In the rest of the text, if there is no risk of confusion, we will write $\mathsf{Rol}_q(X,Y)$ simply as Rol_q.

The covariant derivatives of the Riemann curvature $\hat{R}$ of $\hat{M}$ in the direction of an arbitrary vector fields $\hat{W}$ on $\hat{M}$ are

$$\begin{aligned}
\hat{D}_{\hat{W}}(\hat{\sigma}^1_A) &:= \hat{g}((\hat{\nabla}_{\hat{W}}\hat{R})(AY, \hat{Z}_A)\hat{Z}_A, AY),\\
\hat{D}_{\hat{W}}(\hat{\sigma}^2_A) &:= \hat{g}((\hat{\nabla}_{\hat{W}}\hat{R})(AX, \hat{Z}_A)\hat{Z}_A, AX),\\
\hat{D}_{\hat{W}}(\hat{\sigma}_A) &:= \hat{g}((\hat{\nabla}_{\hat{W}}\hat{R})(AY, AX)AX, AY)\\
\hat{D}_{\hat{W}}(\Pi_X) &:= \hat{g}((\hat{\nabla}_{\hat{W}}\hat{R})(\star \hat{Z}_A), \star AX),\\
\hat{D}_{\hat{W}}(\Pi_Y) &:= \hat{g}((\hat{\nabla}_{\hat{W}}\hat{R})(\star \hat{Z}_A), \star AY),\\
\hat{D}_{\hat{W}}(\Pi_Z) &:= \hat{g}((\hat{\nabla}_{\hat{W}}\hat{R})(\star AX), \star AY).
\end{aligned}$$

Fix the vectors fields $W = aX + bY$ on M such that $a, b \in C^\infty(M)$ and $\hat{W}$ on $\hat{M}$, then

$$\mathscr{L}_{NS}(W,\hat{W})|_q \hat{Z}_A = 0, \tag{5.3}$$
$$\mathscr{L}_R(W)|_q \hat{\sigma}_{(\cdot)} = \hat{D}_{AW}(\hat{\sigma}_A), \tag{5.4}$$
$$\mathscr{L}_R(W)|_q \Pi_Y = \hat{D}_{AW}(\Pi_Y) - (a\Gamma^1_{(1,2)} + b\Gamma^2_{(1,2)})\Pi_X, \tag{5.5}$$
$$\mathscr{L}_R(W)|_q \Pi_X = \hat{D}_{AW}(\Pi_X) + (a\Gamma^1_{(1,2)} + b\Gamma^2_{(1,2)})\Pi_Y. \tag{5.6}$$

Thus we get,

$$\mathscr{L}_R(W)|_q \mathsf{Rol}_{(\cdot)} = -(W(K) - \hat{D}_{AW}(\hat{\sigma}_A))A(X\wedge Y) + \hat{D}_{AW}(\Pi_Y)\theta_X \otimes \hat{Z}_A - \hat{D}_{AW}(\Pi_X)\theta_Y \otimes \hat{Z}_A.$$

Using Proposition 3.1.16 to obtain the next lemma.

Lemma 5.1.1. *The first order Lie brackets between the vector fields on Q are*

$$
\begin{aligned}
&[\mathscr{L}_R(X), \mathscr{L}_R(Y)]|_q = -\gamma^1 \mathscr{L}_R(X)|_q - \gamma^2 \mathscr{L}_R(Y)|_q + \nu(\mathsf{Rol}_q)|_q,\\
&[\mathscr{L}_R(X), \nu((\cdot)(X\wedge Y))]|_q = -\mathscr{L}_{NS}(AY)|_q,\\
&[\mathscr{L}_R(X), \nu(\theta_X \otimes \hat{Z})]|_q = -\mathscr{L}_{NS}(\hat{Z}_A)|_q + \gamma^1 \nu(\theta_Y \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_R(X), \nu(\theta_Y \otimes \hat{Z})]|_q = -\gamma^1 \nu(\theta_X \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_R(X), \mathscr{L}_{NS}((\cdot)X)]|_q = \gamma^1 \mathscr{L}_{NS}(AY)|_q,\\
&[\mathscr{L}_R(X), \mathscr{L}_{NS}((\cdot)Y)]|_q = -\gamma^1 \mathscr{L}_{NS}(AX)|_q + \nu(\hat{\sigma}_A A(X\wedge Y) + \Pi_Y \theta_X \otimes \hat{Z}_A - \Pi_X \theta_Y \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_R(X), \mathscr{L}_{NS}(\hat{Z}_A)]|_q = \nu(\Pi_Y A(X\wedge Y) + \hat{\sigma}^2_A \theta_X \otimes \hat{Z}_A + \Pi_Z \theta_Y \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
&[\mathscr{L}_R(Y), \nu((\cdot)(X\wedge Y))]|_q = \mathscr{L}_{NS}(AX)|_q,\\
&[\mathscr{L}_R(Y), \nu(\theta_X \otimes \hat{Z})]|_q = \gamma^2 \nu(\theta_Y \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_R(Y), \nu(\theta_Y \otimes \hat{Z})]|_q = -\mathscr{L}_{NS}(\hat{Z}_A)|_q - \gamma^2 \nu(\theta_X \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_R(Y), \mathscr{L}_{NS}((\cdot)X)]|_q = \gamma^2 \mathscr{L}_{NS}(AY)|_q - \nu(\hat{\sigma}_A A(X\wedge Y) + \Pi_Y \theta_X \otimes \hat{Z}_A - \Pi_X \theta_Y \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_R(Y), \mathscr{L}_{NS}((\cdot)Y)]|_q = -\gamma^2 \mathscr{L}_{NS}(AX)|_q,\\
&[\mathscr{L}_R(Y), \mathscr{L}_{NS}(\hat{Z}_A)]|_q = \nu(-\Pi_X A(X\wedge Y) + \Pi_Z \theta_X \otimes \hat{Z}_A + \hat{\sigma}^1_A \theta_Y \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
&[\mathscr{L}_{NS}(\hat{Z}), \mathscr{L}_{NS}((\cdot)X)]|_q = -\nu(\Pi_Y A(X\wedge Y) + \hat{\sigma}^2_A \theta_X \otimes \hat{Z}_A + \Pi_Z \theta_Y \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_{NS}(\hat{Z}), \mathscr{L}_{NS}((\cdot)Y)]|_q = -\nu(-\Pi_X A(X\wedge Y) + \Pi_Z \theta_X \otimes \hat{Z}_A + \hat{\sigma}^1_A \theta_Y \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_{NS}(\hat{Z}), \nu((\cdot)X\wedge Y)]|_q = 0,\\
&[\mathscr{L}_{NS}(\hat{Z}), \nu(\theta_X \otimes \hat{Z})]|_q = \mathscr{L}_{NS}(AX)|_q,\\
&[\mathscr{L}_{NS}(\hat{Z}), \nu(\theta_Y \otimes \hat{Z})]|_q = \mathscr{L}_{NS}(AY)|_q,
\end{aligned}
$$

$$
\begin{aligned}
&[\mathscr{L}_{NS}((\cdot)X), \mathscr{L}_{NS}((\cdot)Y)]|_q = \nu(\hat{\sigma}_A A(X\wedge Y) + \Pi_Y \theta_X \otimes \hat{Z}_A - \Pi_X \theta_Y \otimes \hat{Z}_A)|_q,\\
&[\mathscr{L}_{NS}((\cdot)X), \nu((\cdot)X\wedge Y)]|_q = -\mathscr{L}_{NS}(AY)|_q,\\
&[\mathscr{L}_{NS}((\cdot)X), \nu(\theta_X \otimes \hat{Z})]|_q = -\mathscr{L}_{NS}(\hat{Z}_A)|_q,\\
&[\mathscr{L}_{NS}((\cdot)X), \nu(\theta_Y \otimes \hat{Z})]|_q = 0,
\end{aligned}
$$

$$
\begin{aligned}
&[\mathscr{L}_{NS}((\cdot)Y), \nu((\cdot)X \wedge Y)]|_q = \mathscr{L}_{NS}(AX)|_q,\\
&[\mathscr{L}_{NS}((\cdot)Y), \nu(\theta_X \otimes \hat{Z})]|_q = 0,\\
&[\mathscr{L}_{NS}((\cdot)Y), \nu(\theta_Y \otimes \hat{Z})]|_q = -\mathscr{L}_{NS}(\hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
&[\nu((\cdot)X \wedge Y), \nu(\theta_X \otimes \hat{Z})]|_q = \nu(\theta_Y \otimes \hat{Z}_A)|_q,\\
&[\nu((\cdot)X \wedge Y), \nu(\theta_Y \otimes \hat{Z})]|_q = -\nu(\theta_X \otimes \hat{Z}_A)|_q,\\
&[\nu(\theta_X \otimes \hat{Z}), \nu(\theta_Y \otimes \hat{Z})]|_q = \nu(A(X \wedge Y))|_q.
\end{aligned}
$$

Lemma 5.1.2. *The derivative of the different curvatures of $\hat{M}$ with respect to the vertical vector fields are*

$$
\begin{array}{lll}
\nu(A(X\wedge Y))|_q\hat{\sigma}^1_{(\cdot)} = -2\Pi_Z, & \nu(\theta_X\otimes\hat{Z}_A)|_q\hat{\sigma}^1_{(\cdot)} = -2\Pi_X, & \nu(\theta_Y\otimes\hat{Z}_A)|_q\hat{\sigma}^1_{(\cdot)} = 0,\\
\nu(A(X\wedge Y))|_q\hat{\sigma}^2_{(\cdot)} = 2\Pi_Z, & \nu(\theta_X\otimes\hat{Z}_A)|_q\hat{\sigma}^2_{(\cdot)} = 0, & \nu(\theta_Y\otimes\hat{Z}_A)|_q\hat{\sigma}^2_{(\cdot)} = -2\Pi_Y,\\
\nu(A(X\wedge Y))|_q\hat{\sigma}_{(\cdot)} = 0, & \nu(\theta_X\otimes\hat{Z}_A)|_q\hat{\sigma}_{(\cdot)} = 2\Pi_X, & \nu(\theta_Y\otimes\hat{Z}_A)|_q\hat{\sigma}_{(\cdot)} = 2\Pi_Y,\\
\nu(A(X\wedge Y))|_q\Pi_X = \Pi_Y, & \nu(\theta_X\otimes\hat{Z}_A)|_q\Pi_X = \hat{\sigma}^1_A - \hat{\sigma}_A, & \nu(\theta_Y\otimes\hat{Z}_A)|_q\Pi_X = -\Pi_Z,\\
\nu(A(X\wedge Y))|_q\Pi_Y = -\Pi_X, & \nu(\theta_X\otimes\hat{Z}_A)|_q\Pi_Y = -\Pi_Z, & \nu(\theta_Y\otimes\hat{Z}_A)|_q\Pi_Y = \hat{\sigma}^2_A - \hat{\sigma}_A,\\
\nu(A(X\wedge Y))|_q\Pi_Z = \hat{\sigma}^1_A - \hat{\sigma}^2_A, & \nu(\theta_X\otimes\hat{Z}_A)|_q\Pi_Z = \Pi_Y, & \nu(\theta_Y\otimes\hat{Z}_A)|_q\Pi_Z = \Pi_X.
\end{array}
$$

Proof. We only need to know the derivatives of AX, AY and $\hat{Z}_A$ with respect to these vertical vectors fields. Indeed, we have

$$
\begin{array}{lll}
\nu(A(X\wedge Y))|_q(\cdot)X = AY, & \nu(\theta_X\otimes\hat{Z}_A)|_q(\cdot)X = \hat{Z}_A, & \nu(\theta_Y\otimes\hat{Z}_A)|_q(\cdot)X = 0,\\
\nu(A(X\wedge Y))|_q(\cdot)Y = -AX, & \nu(\theta_X\otimes\hat{Z}_A)|_q(\cdot)Y = 0, & \nu(\theta_Y\otimes\hat{Z}_A)|_q(\cdot)Y = \hat{Z}_A,\\
\nu(A(X\wedge Y))|_q\hat{Z} = 0, & \nu(\theta_X\otimes\hat{Z}_A)|_q\hat{Z} = -AX, & \nu(\theta_Y\otimes\hat{Z}_A)|_q\hat{Z} = -AY.
\end{array}
$$

□

We next proceed to find the necessary conditions for the non-controllable situation of the rolling system of M against $\hat{M}$. Then, we will assume that an orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ of some q_0 is not open in Q. Thus, we have

Theorem 5.1.3. *Let (M,g) and $(\hat{M},\hat{g})$ be two connected Riemannian manifolds of dimensions 2 and 3 respectively. Assume that, for a $q_0 = (x_0,\hat{x}_0;A_0) \in Q$, the orbit $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is not open in Q, then,*

1. *If $K - \hat{\sigma}_A = 0$ on an open set of $q = (x,\hat{x};A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ and $(\Pi_X,\Pi_Y) = 0$ on a neighbourhood of $\hat{x}$, then $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 2$ and M is isometric to a totally geodesic submanifold of $\hat{M}$.*

2. *If $K - \hat{\sigma}_A \neq 0$ on an open set of $q = (x,\hat{x};A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ and $(\Pi_X,\Pi_Y) = 0$ on a neighbourhood of $\hat{x}$, then $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 5$ and $\hat{M}$ is a warped product of real interval with 2-dimensional totally geodesic submanifold.*

3. *If $K - \hat{\sigma}_A = 0$ on an open set of $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ and $(\Pi_X, \Pi_Y) \neq 0$ on a neighbourhood of $\hat{x}$, then $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 7$, M has constant curvature and $\hat{M}$ is (locally) a Riemannian product of real interval with 2-dimensional Riemannian submanifold.*

4. *If $K - \hat{\sigma}_A \neq 0$ on an open set of $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ et $(\Pi_X, \Pi_Y) \neq 0$ on a neighbourhood of $\hat{x}$, then the dimension of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is equal to 5 or 6 or 7. In particular, if $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 5$ then M is a flat manifold and $\hat{M}$ is (locally) a Riemannian product of real interval with 2-dimensional totally geodesic Riemannian submanifold.*

Remark 5.1.4. Up to the date of writing of the current text, there have been no result about the geometrical aspect of M and $\hat{M}$ when $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 6$ and when $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 7$ in the last case of the previous theorem.

The aim of this chapter is to prove Theorem 5.1.3 by distinguishing two cases: when $\Pi_X = \Pi_Y \equiv 0$ and when $(\Pi_X, \Pi_Y) \neq (0, 0)$ on a neighbourhood $\hat{V}$ of $\hat{x} = \pi_{Q,\hat{M}}(q)$ where $q \in \mathcal{O}_{\mathscr{D}_R}(q_0)$.

5.2 Case $(\Pi_X, \Pi_Y) = (0, 0)$

The section is decomposed to two parts, the first part is when $\hat{\sigma}_A \equiv K(x)$ in an open set of a $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ and the second part is when $\hat{\sigma}_A \neq K(x)$ in an open set of q.

5.2.1 Case $(\Pi_X, \Pi_Y) = (0, 0)$ and $\hat{\sigma}_A \equiv K(x)$

Theorem 5.2.1. *Under the assumption that $(\Pi_X, \Pi_Y) = (0, 0)$ on $\hat{V}$, we suppose in addition that $\hat{\sigma}_A \equiv K(x)$ in an open set of $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$. Then, the image of the g-geodesics of M by $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),\hat{M}} \circ \pi^{-1}_{\mathcal{O}_{\mathscr{D}_R}(q_0),M}$ are also $\hat{g}$-geodesics of $\hat{M}$.*

Proof. Let $O = (\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),\hat{M}})^{-1}(\hat{V})$ be an open set in Q. Since, $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),\hat{M}}$ is a submersion then we have,

$$\mathsf{Rol}_q(X, Y) \equiv 0, \quad \forall q \in O \cap \mathcal{O}_{\mathscr{D}_R}(q_0).$$

By Proposition 4.2.9, $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is a 2-dimensional integral manifold of $\mathscr{D}_R$ and $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),M}$ is a Riemannian covering map. Thus, $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),M}$ is a local isometry. This implies that the reciprocal image of a g-geodesic by $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),M}$ is also a geodesic of $\mathcal{O}_{\mathscr{D}_R}(q_0)$. The mentioned proposition also proved that $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),\hat{M}}$ is a Riemannian immersion which maps the geodesics of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ to $\hat{g}$-geodesics. $\square$

5.2.2 Case $(\Pi_X, \Pi_Y) = (0,0)$ and $\hat{\sigma}_A \neq K(x)$

Proposition 5.2.2. *Assume that $(\Pi_X, \Pi_Y) = (0,0)$ on $\hat{V}$ and $\hat{\sigma}_A - K(x)$ is not vanish on an open set of $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, then we have $dim\mathcal{O}_{\mathscr{D}_R}(q_0) = 5$.*

Proof. With respect to the assumptions announced in the proposition, we have,

$$\mathsf{Rol}_q(X, Y) = (K - \hat{\sigma}_A)\nu(A(X \wedge Y))|_q, \forall q = (x, \hat{x}; A) \in O \cap \mathcal{O}_{\mathscr{D}_R}(q_0),$$

where $O := \pi^{-1}_{Q,\hat{M}}(\hat{V})$. By using Lemma 5.1.1, $[\mathscr{L}_R(X), \nu((\cdot)(X\wedge Y))]|_q$ and $[\mathscr{L}_R(Y), \nu((\cdot)(X \wedge Y))]|_q$ imply that $\mathscr{L}_{NS}(AX)|_q$ and $\mathscr{L}_{NS}(AY)|_q$ are tangent to $O_1 := O \cap \mathcal{O}_{\mathscr{D}_R}(q_0)$. Thus, the linearly independent vector fields

$$\mathscr{L}_R(X)|_q, \mathscr{L}_R(Y)|_q, \nu(A(X \wedge Y))|_q, \mathscr{L}_{NS}(AX)|_q, \mathscr{L}_{NS}(AY)|_q,$$

generate the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ at $q \in O_1$. This completes the proof. □

Corollary 5.2.3. *Under our essential assumptions of Proposition 5.2.2, there is a locally oriented orthonormal frame $\hat{E}_1, \hat{E}_2, \hat{E}_3$ defined on $\hat{V}$ with respect to which the connection table are in the form*

$$\hat{\Gamma} = \begin{pmatrix} 0 & 0 & \hat{\Gamma}^3_{(2,3)} \\ 0 & 0 & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & \hat{\Gamma}^2_{(1,2)} & \hat{\Gamma}^3_{(1,2)} \end{pmatrix}. \tag{5.7}$$

Proof. The matrix corresponding to $\hat{R}$ with respect to the basis $\star AX$, $\star AY$, $\star\hat{Z}_A$ is

$$\begin{pmatrix} -\hat{\sigma}^1_A & \Pi_{\hat{Z}} & 0 \\ \Pi_{\hat{Z}} & -\hat{\sigma}^2_A & 0 \\ 0 & 0 & -\hat{\sigma}_A \end{pmatrix}.$$

Then, $-\hat{\sigma}_A$ is an eigenvalue of $\hat{R}$ with the corresponding eigenvector $\star\hat{Z}_A$, we denote the other two eigenvalues of $\hat{R}$ by $-\hat{K}_1$ and $-\hat{K}_2$.
Since the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ at $q = (x, \hat{x}; A) \in O_1$ is generated by $\mathscr{L}_{NS}(X)|_q$, $\mathscr{L}_{NS}(Y)|_q$, $\nu(A(X \wedge Y))|_q$, $\mathscr{L}_{NS}(AX)|_q$, $\mathscr{L}_{NS}(AY)|_q$, then the image of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ by $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),\hat{M}}$ is (locally) a 2-dimensional smooth submanifold of $\hat{M}$ which we called $\hat{N}$.
The tangent space of $\hat{N}$ is generated by $\{AX, AY\}$. These vectors are defined on the neighbourhood $\hat{V}$ of $\hat{x}$ in $\hat{M}$. The normal vector of $\hat{N}$ on $\hat{V}_1 := \hat{V} \cap \hat{N}$ is $\hat{Z}_A$ which depend only of $\hat{x}$. One can extend this normal vector to another one $\hat{E}_3|\hat{x}$ defined on $\hat{V}$. Then we can determine (locally on $\hat{V}$) two orthonormal vectors $\hat{E}_1|\hat{x}, \hat{E}_2|\hat{x}$ orthogonal to $\hat{E}_3|\hat{x}$. Thus, $\{\hat{E}_1, \hat{E}_2, \hat{E}_3\}$ is a locally orthonormal frame of $(\hat{M}, \hat{g})$ where $\hat{N}$ is generated by $\hat{E}_1, \hat{E}_2$ on $\hat{V}_1$. Since $\hat{E}_3 = \hat{Z}_A$ and $span\{\hat{E}_1, \hat{E}_2\} = span\{AX, AY\}$ on $\hat{V}_1$, then we must have that $\star\hat{E}_1, \star\hat{E}_2, \star\hat{E}_3$ are the eigenvectors of the eigenvalues $-\hat{K}_1$, $-\hat{K}_2$, $-\hat{\sigma}_A$ of $\hat{R}$. Because $\nu(A(X \wedge Y)$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$ then so is also $\nu(\star\hat{E}_3 A)|_q$. Moreover,

we have $\mathscr{L}_{NS}(\hat{E}_1)|_q$ and $\mathscr{L}_{NS}(\hat{E}_2)|_q$ are tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. These informations now push us to define the smooth function $\psi : \mathcal{O}_{\mathscr{D}_R}(q_0) \to \mathbb{R}$ such that,

$$AX = cos(\psi(q))\hat{E}_1 + sin(\psi(q))\hat{E}_2, \tag{5.8}$$

$$AY = -sin(\psi(q))\hat{E}_1 + cos(\psi(q))\hat{E}_2. \tag{5.9}$$

We begin by computing in the tangent space of O_1,

$$[\mathscr{L}_{NS}(\hat{E}_1), \nu(\star\hat{E}_3(\cdot))]|_q = \nu(\star(\hat{\Gamma}^1_{(3,1)}\hat{E}_1 - \hat{\Gamma}^1_{(2,3)}\hat{E}_2)A)|_q := F_1|_q,$$

therefore, $F_1|_q$ is also a vector field on $\mathcal{O}_{\mathscr{D}_R}(q_0)$ at $q \in O_1$. We continue computing

$$[F_1, \nu(\star\hat{E}_3(\cdot))]|_q = \nu(\star(-\hat{\Gamma}^1_{(3,1)}\hat{E}_2 + \hat{\Gamma}^1_{(2,3)}\hat{E}_1)A)|_q := F_2|_q,$$

then $F_2|_q$ is also a vector field of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ at $q \in O_1$. Now, if $\nu(\star\hat{E}_3A)|_q, F_1|_q, F_2|_q$ are linearly independent for $q \in O_1$, it would follow that they form a basis of $V|_q(\pi_Q)$ for $q \in O_1$ and hence $V|_q(\pi_Q) \subset T_q(\mathcal{O}_{\mathscr{D}_R}(q_0))$ for $q \in O_1$. Then Corollary 5.6 in [28] would imply that $(\Sigma)_R$ is completely controllable and $\mathcal{O}_{\mathscr{D}_R}(q_0)$ will be of dimension 8, which is a contradiction. Hence, at least in a dense open subset O_2 of $\mathcal{O}_{\mathscr{D}_R}(q_0)$, one has that $\nu(\star\hat{E}_2A)|_q, F_1|_q, F_2|_q$ are linearly dependent i.e.

$$0 = det\begin{pmatrix} 0 & 1 & 0 \\ -\hat{\Gamma}^1_{(3,1)} & 0 & \hat{\Gamma}^1_{(2,3)} \\ -\hat{\Gamma}^1_{(2,3)} & 0 & -\hat{\Gamma}^1_{(3,1)} \end{pmatrix} = -((\hat{\Gamma}^1_{(3,1)})^2 + (\hat{\Gamma}^1_{(2,3)})^2).$$

Thus $\hat{\Gamma}^1_{(3,1)} = 0$ and $\hat{\Gamma}^1_{(2,3)} = 0$ on $\hat{V}_2 := \hat{V}_1 \cup \pi^{-1}_{\mathcal{O}_{\mathscr{D}_R}(q_0),\hat{M}}(O_2)$.
Repeat the above method with the Lie bracket $[\mathscr{L}_{NS}(\hat{E}_2), \nu(\star\hat{E}_3(\cdot))]|_q$, we found that

$$\hat{\Gamma}^2_{(3,1)} = 0 \; and \; \hat{\Gamma}^2_{(2,3)} = 0.$$

Therefore, the connection of $\hat{M}$ (locally) has the matrix form

$$\begin{pmatrix} 0 & 0 & \hat{\Gamma}^3_{(2,3)} \\ 0 & 0 & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & \hat{\Gamma}^2_{(1,2)} & \hat{\Gamma}^3_{(1,2)} \end{pmatrix}.$$

□

Next, we will conclude some results about the geometrical aspect of $\hat{M}$.

Corollary 5.2.4. *With the assumptions of Proposition 5.2.2$\hat{M}$ contains a 2-dimensional, totally geodesic, submanifold.*

Proof. By Lemma 5.1.1, the Lie brackets between $\mathscr{L}_{NS}(AX)|_q, \mathscr{L}_{NS}(AY)|_q, \nu(A(X \wedge Y))|_q$ form a Lie sub-algebra of the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Using Equations (5.5) and (5.6), one can easily deduced that $\hat{D}_{AX}(\Pi_X) \equiv \hat{D}_{AY}(\Pi_X) \equiv \hat{D}_{AX}(\Pi_Y) \equiv \hat{D}_{AY}(\Pi_Y) \equiv 0$ and hence $(\nabla_{\hat{E}_i}\hat{R})(\star\hat{Z}_A) = 0$ for $i = 1, 2$. Then, $\hat{M}$ can be reduced to the warped product of $\hat{Z}_A$ with the 2-dimensional, totally geodesic, submanifold $\hat{N}$ generated by the two vectors $\hat{E}_1$ and $\hat{E}_2$ defined in Corollary 5.2.3. □

We will establish the sufficient condition of Corollary 5.2.4 in the following corollary.

Corollary 5.2.5. *Suppose that $\hat{M}$ continue a 2-dimensional, totally geodesic, submanifold $\hat{N}$, then* $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 5$.

Proof. Suppose that $\hat{N}$, the 2-dimensional totally geodesic submanifold of M, is generated by the two vector fields $\hat{E}_1$, $\hat{E}_2$ given in Corollary 5.2.3. Fix some local orthonormal frame X, Y of M. We will proof that the coefficients of AX, AY with respect to $\hat{Z}_A$ are zero and hence we stay in the same situation i.e. $\hat{M}$ locally is a Riemannian product of $\hat{N}$ with $\hat{Z}_A$. We have

$$AX = A_1^1\hat{E}_1 + A_1^2\hat{E}_2 + A_1^3\hat{Z}_A,$$
$$AY = A_2^1\hat{E}_1 + A_2^2\hat{E}_2 + A_2^3\hat{Z}_A,$$

where (A_j^i) is the i-th row and j-th column of the matrix represented A with respect to the frame X, Y of M and to the frame $\hat{E}_1$, $\hat{E}_2$, $\hat{Z}_A$ of $\hat{M}$. Further more, by Appendix A in [10], if $t \mapsto \gamma(t)$, $t \mapsto \hat{\gamma}(t)$ be two smooth curves in a neighbourhoods of x_0 and $\hat{x}_0$ in M and $\hat{M}$ respectively such that $\gamma(0) = x_0$, $\hat{\gamma}(0) = \hat{x}_0$, then $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ is a rolling curve if and only if

$$\begin{cases} \dot{\gamma}(t) = \sum_i v^i(t) X_i|_{\gamma(t)}, \\ \\ \dot{\hat{\gamma}}(t) = \sum_i v^i(t) A(t) X_i|_{\gamma(t)}, \\ \\ \frac{dA_j^i}{dt}(t) = \sum_m \big(\sum_k \Gamma_{(j,k)}^m(\gamma(t)) A_k^i(t) - \sum_{k,l} \hat{\Gamma}_{(k,i)}^l(\hat{\gamma}(t)) A_j^k(t) A_m^l(t) \big) v^m(t). \end{cases}$$

We are interested in the values of A_1^3 and A_2^3 such we have

$$\frac{dA_1^3}{dt}(t) = (\gamma^1 A_2^3 + \hat{\Gamma}_{(3,1)}^3 A_1^1 A_1^3 - \hat{\Gamma}_{(2,3)}^3 A_1^2 A_1^3) v^1 + (\gamma^2 A_2^3 + \hat{\Gamma}_{(3,1)}^3 A_1^1 A_2^3 - \hat{\Gamma}_{(2,3)}^3 A_1^2 A_2^3) v^2,$$
$$\frac{dA_2^3}{dt}(t) = (-\gamma^1 A_1^3 + \hat{\Gamma}_{(3,1)}^3 A_2^1 A_1^3 - \hat{\Gamma}_{(2,3)}^3 A_2^2 A_1^3) v^1 + (-\gamma^2 A_1^3 + \hat{\Gamma}_{(3,1)}^3 A_2^1 A_2^3 - \hat{\Gamma}_{(2,3)}^3 A_2^2 A_2^3) v^2.$$

In this situation , one can write the previous two equalities in the following form

$$\begin{pmatrix} \frac{dA_1^3}{dt}(t) \\ \\ \frac{dA_2^3}{dt}(t) \end{pmatrix} = B(t) \begin{pmatrix} A_1^3(t) \\ \\ A_2^3(t) \end{pmatrix},$$

where $B(t)$ is the matrix 2×2

$$\begin{pmatrix} (\hat{\Gamma}_{(3,1)}^3 A_1^1 - \hat{\Gamma}_{(2,3)}^3 A_1^2) v^1 & \gamma^1 v^1 + (\gamma^2 + \hat{\Gamma}_{(3,1)}^3 A_1^1 - \hat{\Gamma}_{(2,3)}^3 A_1^2) v^2 \\ \\ (-\gamma^1 + \hat{\Gamma}_{(3,1)}^3 A_2^1 - \hat{\Gamma}_{(2,3)}^3 A_2^2) v^1 - \gamma^2 v^2 & (\hat{\Gamma}_{(3,1)}^3 A_2^1 - \hat{\Gamma}_{(2,3)}^3 A_2^2) v^2 \end{pmatrix}.$$

If the initial value of this problem is 0, when $t = 0$, by the uniqueness of solution of Cauchy problem, we get $A_1^3 = A_2^3 = 0$. Then, $span\{AX, AY\} = span\{\hat{E}_1, \hat{E}_2\}$. So, one can repeat the same calculation of the proof of Proposition 5.2.2 and get that $dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 5$. ☐

5.3 Case $(\Pi_X, \Pi_Y) \neq (0,0)$

In this section, in addition to the preliminary notations, we will introduce a new frame in the computation by rotating the fixed frame X, Y of M by an angle. Indeed, since we here assume that, for every $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, there exists a neighbourhood $\hat{V}$ in $\hat{M}$ where (Π_X, Π_Y) is not vanish on $\hat{V}$, then this push us to define the smooth functions r, $\phi : Q \to \mathbb{R}$ in a small enough open neighbourhood (we may shrink $(\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),\hat{M}})^{-1}(\hat{V}) := O$) by

$$\begin{aligned} \Pi_X &= rcos\phi, \\ \Pi_Y &= rsin\phi. \end{aligned} \tag{5.10}$$

In order to simplify the notation, we write $c_\phi := cos\phi$ and $s_\phi := sin\phi$. Then, we define the "Q-dependent" vector fields

$$\begin{aligned} \tilde{X}_A &= c_\phi X + s_\phi Y, \\ \tilde{Y}_A &= -s_\phi X + c_\phi Y. \end{aligned} \tag{5.11}$$

Using Equation (5.10), we get

$$\Pi_{\tilde{X}} := \hat{g}(\hat{R}(\star \hat{Z}_A), \star A\tilde{X}_A) = c_\phi \Pi_X + s_\phi \Pi_Y = r, \tag{5.12}$$

$$\Pi_{\tilde{Y}} := \hat{g}(\hat{R}(\star \hat{Z}_A), \star A\tilde{Y}_A) = -s_\phi \Pi_X + c_\phi \Pi_Y = 0. \tag{5.13}$$

With respect to the new frame, we have

$$\begin{aligned} \tilde{\hat{\sigma}}^1_A &:= -\hat{g}(\hat{R}(\star A\tilde{X}_A), \star A\tilde{X}_A) = c^2_\phi \hat{\sigma}^1_A + s^2_\phi \hat{\sigma}^2_A - 2c_\phi s_\phi \Pi_Z, \\ \tilde{\hat{\sigma}}^2_A &:= -\hat{g}(\hat{R}(\star A\tilde{Y}_A), \star A\tilde{Y}_A) = s^2_\phi \hat{\sigma}^1_A + c^2_\phi \hat{\sigma}^2_A + 2c_\phi s_\phi \Pi_Z, \\ \tilde{\Pi}_{\hat{Z}} &:= \hat{g}(\hat{R}(\star A\tilde{X}_A), \star A\tilde{Y}_A) = s_\phi c_\phi (\hat{\sigma}^1_A - \hat{\sigma}^2_A) + (c^2_\phi - s^2_\phi)\Pi_Z. \end{aligned}$$

Similarly to Lemma 5.1.1, we introduce the next lemma.

Lemma 5.3.1. *The first order Lie brackets on Q regarding the new frame are*

$$[\mathscr{L}_R(\tilde{X}), \nu((\cdot)(X \wedge Y))]|_q = \mathscr{L}_R(\tilde{Y}_A)|_q - \mathscr{L}_{NS}(A\tilde{Y}_A)|_q,$$

$$\begin{aligned} & [\mathscr{L}_R(\tilde{X}), \nu(\theta_{\tilde{X}} \otimes \hat{Z})]|_q \\ = \; & -(\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q - \mathscr{L}_{NS}(\hat{Z}_A)|_q + (\mathscr{L}_R(\tilde{X}_A)|_q\phi + g(\Gamma, \tilde{X}_A))\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q, \end{aligned}$$

$$\begin{aligned} & [\mathscr{L}_R(\tilde{X}), \nu(\theta_{\tilde{Y}} \otimes \hat{Z})]|_q \\ = \; & -(\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q - (\mathscr{L}_R(\tilde{X}_A)|_q\phi + g(\Gamma, \tilde{X}_A))\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q, \end{aligned}$$

$$\begin{aligned} & [\mathscr{L}_R(\tilde{X}), \mathscr{L}_{NS}((\cdot)\tilde{X})]|_q \\ = \; & -(\mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q + (\mathscr{L}_R(\tilde{X}_A)|_q\phi + g(\Gamma, \tilde{X}_A))\mathscr{L}_{NS}(A\tilde{Y}_A)|_q, \end{aligned}$$

$$\begin{aligned} & [\mathscr{L}_R(\tilde{X}), \mathscr{L}_{NS}((\cdot)\tilde{Y})]|_q \\ = \; & -(\mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q - (\mathscr{L}_R(\tilde{X}_A)|_q\phi + g(\Gamma, \tilde{X}_A))\mathscr{L}_{NS}(A\tilde{X}_A)|_q + \nu(\mathsf{Rol}_q)|_q \\ & +K\nu(A(X \wedge Y))|_q, \end{aligned}$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{X}), \mathscr{L}_{NS}(\hat{Z})]|_q \\
= \ & -(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q + \tilde{\tilde{\sigma}}_A^2 \nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q + \tilde{\Pi}_{\hat{Z}}\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
[\mathscr{L}_R(\tilde{Y}), \nu((\cdot)(X \wedge Y))]|_q = -\mathscr{L}_R(\tilde{X}_A)|_q + \mathscr{L}_{NS}(A\tilde{X}_A)|_q,
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), \nu(\theta_{\tilde{X}} \otimes \hat{Z})]|_q \\
= \ & (\mathscr{L}_R(\tilde{Y}_A)|_q\phi + g(\Gamma, \tilde{Y}_A))\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q + (\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), \nu(\theta_{\tilde{Y}} \otimes \hat{Z})]|_q \\
= \ & -(\mathscr{L}_R(\tilde{Y}_A)|_q\phi - \mathscr{L}_{NS}(\hat{Z}_A)|_q + g(\Gamma, \tilde{Y}_A))\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q + (\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), \mathscr{L}_{NS}((\cdot)\tilde{X})]|_q \\
= \ & (\mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + (\mathscr{L}_R(\tilde{Y}_A)|_q\phi + g(\Gamma, \tilde{Y}_A))\mathscr{L}_{NS}(A\tilde{Y}_A)|_q - \nu(\mathsf{Rol}_q)|_q \\
& -K\nu(A(X \wedge Y))|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), \mathscr{L}_{NS}((\cdot)\tilde{Y})]|_q \\
= \ & (\mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q - (\mathscr{L}_R(\tilde{Y}_A)|_q\phi + g(\Gamma, \tilde{Y}_A))\mathscr{L}_{NS}(A\tilde{X}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), \mathscr{L}_{NS}(\hat{Z})]|_q \\
= \ & (\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + \tilde{\Pi}_{\hat{Z}}\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q + \tilde{\tilde{\sigma}}_A^1\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q - \Pi_{\tilde{X}}\nu(A(X \wedge Y))|_q,
\end{aligned}
$$

$$
[\nu((\cdot)X \wedge Y), \nu(\theta_{\tilde{Y}} \otimes \hat{Z})]|_q = 0,
$$

$$
[\nu((\cdot)X \wedge Y), \nu(\theta_{\tilde{X}} \otimes \hat{Z})]|_q = 0,
$$

$$
[\nu((\cdot)X \wedge Y), \mathscr{L}_{NS}((\cdot)\tilde{X})]|_q = 0,
$$

$$
[\nu((\cdot)X \wedge Y), \mathscr{L}_{NS}((\cdot)\tilde{Y})]|_q = 0,
$$

$$
[\nu((\cdot)X \wedge Y), \mathscr{L}_{NS}(\hat{Z})]|_q = 0,
$$

$$
\begin{aligned}
& [\nu(\theta_{\tilde{X}} \otimes \hat{Z}), \nu(\theta_{\tilde{Y}} \otimes \hat{Z})] \\
= \ & -(\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q - (\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q + \nu(A(X \wedge Y))|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\nu(\theta_{\tilde{X}} \otimes \hat{Z}), \mathscr{L}_{NS}((\cdot)\tilde{X})]|_q \\
= \ & (\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{Y}_A)|_q - (\mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q + \mathscr{L}_{NS}(\hat{Z}_A)|_q,
\end{aligned}
$$

$$\begin{aligned}
& [\nu(\theta_{\tilde{X}}\otimes\hat{Z}),\mathscr{L}_{NS}((\cdot)\tilde{Y})]|_q \\
= \ & -(\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{X}_A)|_q-(\mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$[\nu(\theta_{\tilde{X}}\otimes\hat{Z}),\mathscr{L}_{NS}(\hat{Z})]|_q=-(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q-\mathscr{L}_{NS}(A\tilde{X}_A)|_q,$$

$$\begin{aligned}
& [\nu(\theta_{\tilde{Y}}\otimes\hat{Z}),\mathscr{L}_{NS}((\cdot)\tilde{X})]|_q \\
= \ & (\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{Y}_A)|_q+(\mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi)\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$\begin{aligned}
& [\nu(\theta_{\tilde{Y}}\otimes\hat{Z}),\mathscr{L}_{NS}((\cdot)\tilde{Y})]|_q \\
= \ & -(\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{X}_A)|_q+(\mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi)\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q+\mathscr{L}_{NS}(\hat{Z}_A)|_q,
\end{aligned}$$

$$[\nu(\theta_{\tilde{Y}}\otimes\hat{Z}),\mathscr{L}_{NS}(\hat{Z})]|_q=-\mathscr{L}_{NS}(A\tilde{Y}_A)|_q+(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q,$$

$$\begin{aligned}
& [\mathscr{L}_{NS}(\hat{Z}),\mathscr{L}_{NS}((\cdot)\tilde{X})]|_q \\
= \ & (\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{Y}_A)|_q-\tilde{\hat{\sigma}}_A^2\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q-\tilde{\Pi}_{\hat{Z}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$\begin{aligned}
& [\mathscr{L}_{NS}(\hat{Z}),\mathscr{L}_{NS}((\cdot)\tilde{Y})]|_q \\
= \ & -(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{X}_A)|_q+\Pi_{\tilde{X}}\nu(A(X\wedge Y))|_q-\tilde{\Pi}_{\hat{Z}}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
& -\tilde{\hat{\sigma}}_A^1\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$[\mathscr{L}_{NS}(\tilde{X}),\mathscr{L}_{NS}(\tilde{Y})]=-K\nu(A(X\wedge Y))|_q,$$

$$\begin{aligned}
& [\mathscr{L}_{NS}((\cdot)\tilde{X}),\mathscr{L}_{NS}((\cdot)\tilde{Y})] \\
= \ & -(\mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{X}_A)|_q-(\mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{Y}_A)|_q+\nu(\mathsf{Rol}_q)|_q \\
& +K\nu(A(X\wedge Y))|_q.
\end{aligned}$$

5.3.1 Case $(\Pi_X,\Pi_Y)\neq(0,0)$ and $\hat{\sigma}_A\equiv K(x)$

Since (Π_X,Π_Y) is not vanish on $\hat{V}$ and $K(x)-\hat{\sigma}_A$ vanish on an open set of $q=(x,\hat{x};A)\in\mathcal{O}_{\mathscr{D}_R}(q_0)$, Rol_q is evidently simplified to

$$\mathsf{Rol}_q=(\Pi_Y\theta_X-\Pi_X\theta_Y)\otimes\hat{Z}_A.$$

With respect to the new frame, it is equal to

$$\mathsf{Rol}_q=-r\theta_{\tilde{Y}_A}\otimes\hat{Z}_A,\forall q.$$

We also denote along this subsection,

$$\beta(q):=\hat{g}((\hat{\nabla}_{\hat{Z}_A}\hat{R})(\star\hat{Z}_A),\star\hat{Z}_A).$$

From Lemma 5.3.1, we can extract the following Lie brackets,

$$
\begin{aligned}
\tilde{L}_{\tilde{X}}|_q : &= [\mathscr{L}_R(\tilde{X}), \nu(\theta_{\tilde{Y}} \otimes \hat{Z})]|_q \\
&= -(\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q + (-\mathscr{L}_R(\tilde{X}_A)|_q\phi - g(\Gamma, \tilde{X}_A))\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q, \\
\tilde{L}_{\tilde{Y}}|_q : &= [\mathscr{L}_R(\tilde{Y}), \nu(\theta_{\tilde{Y}} \otimes \hat{Z})]|_q \\
&= (\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q - (\mathscr{L}_R(\tilde{Y}_A)|_q\phi + g(\Gamma, \tilde{Y}_A))\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
&\quad -\mathscr{L}_{NS}(\hat{Z}_A)|_q.
\end{aligned}
\tag{5.14}
$$

We can conclude $\mathscr{L}_R(\tilde{X}_A)|_q\phi$ and $\mathscr{L}_R(\tilde{Y}_A)|_q\phi$ from the derivatives of $\hat{\sigma}_A$ and K with respect to the two brackets in (5.14). Indeed,

$$\tilde{L}_{\tilde{X}}|_q\hat{\sigma}_{(\cdot)} = -(\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q\hat{\sigma}_{(\cdot)} + (-\mathscr{L}_R(\tilde{X}_A)|_q\phi - g(\Gamma, \tilde{X}_A))2r,$$

$$\tilde{L}_{\tilde{Y}}|_q\hat{\sigma}_{(\cdot)} = +(\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q\hat{\sigma}_{(\cdot)} + \beta(q) - (\mathscr{L}_R(\tilde{Y}_A)|_q\phi + g(\Gamma, \tilde{Y}_A))2r,$$

and,

$$\tilde{L}_{\tilde{X}}|_qK = -(\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_qK,$$

$$\tilde{L}_{\tilde{Y}}|_qK = +(\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_qK.$$

Thus, since $r \neq 0$ and $K \equiv \hat{\sigma}_A$, we deduce that

$$\mathscr{L}_R(\tilde{X}_A)|_q\phi = -g(\Gamma, \tilde{X}_A), \tag{5.15}$$

$$\mathscr{L}_R(\tilde{Y}_A)|_q\phi = -g(\Gamma, \tilde{Y}_A) + \frac{1}{2r}\beta(q). \tag{5.16}$$

Using (5.15) and (5.16), we obtain,

$$\mathscr{L}_R(\tilde{X}_A)|_q\tilde{X} = 0, \qquad \mathscr{L}_R(\tilde{X}_A)|_q\tilde{Y} = 0,$$

$$\mathscr{L}_R(\tilde{Y}_A)|_q\tilde{X} = \tfrac{1}{2r}\beta(q)\tilde{Y}_A, \quad \mathscr{L}_R(\tilde{Y}_A)|_q\tilde{Y} = -\tfrac{1}{2r}\beta(q)\tilde{X}_A.$$

We next need to use the derivatives of $\hat{\sigma}_A$, r, ϕ, $\frac{\beta(\cdot)}{2r}$, $\tilde{\hat{\sigma}}^1_{(\cdot)}$, $\tilde{\hat{\sigma}}^2_{(\cdot)}$, $\tilde{\Pi}_{\hat{Z}}$ as a functions of $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$. We show them in the following six lemmas.

Lemma 5.3.2. *We have the following derivatives of $\hat{\sigma}_{(\cdot)}$:*

$$
\begin{aligned}
&\mathscr{L}_{NS}(\tilde{X}_A)|_q\hat{\sigma}_{(\cdot)} = 0, && \mathscr{L}_{NS}(\tilde{Y}_A)|_q\hat{\sigma}_{(\cdot)} = 0,\\
&\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\hat{\sigma}_{(\cdot)} = 0, && \nu(A(X \wedge Y))|_q\hat{\sigma}_{(\cdot)} = 0,\\
&\mathscr{L}_{NS}(\hat{Z}_A)|_q\hat{\sigma}_{(\cdot)} = -\beta(q), && \nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\hat{\sigma}_{(\cdot)} = 2r.
\end{aligned}
$$

Lemma 5.3.3. *We have the following derivatives of r:*

$$
\begin{aligned}
&\mathscr{L}_R(\tilde{X}_A)|_qr = -\frac{3}{2}\beta(q), && \mathscr{L}_{NS}(A\tilde{X}_A)|_qr = -\beta(q),\\
&\mathscr{L}_{NS}(\tilde{X}_A)|_qr = -\frac{\beta(q)}{2}, && \mathscr{L}_{NS}(\tilde{Y}_A)|_qr = 0,\\
&\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_qr = -\tilde{\Pi}_{\hat{Z}}, && \nu(A(X \wedge Y))|_qr = 0,\\
&\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_qr = \tilde{\hat{\sigma}}^1_A - \hat{\sigma}_A.
\end{aligned}
$$

Lemma 5.3.4. *We have the following derivatives of ϕ:*

$$\mathscr{L}_{NS}(\tilde{X}_A)|_q\phi = -g(\Gamma, \tilde{X}_A), \qquad \mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi = 0,$$
$$\mathscr{L}_{NS}(\tilde{Y}_A)|_q\phi = -g(\Gamma, \tilde{Y}_A), \qquad \mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi = \frac{\beta(q)}{2r},$$
$$\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi = \frac{1}{r}(\tilde{\hat{\sigma}}^2_A - \hat{\sigma}_A), \qquad \nu(A(X \wedge Y))|_q\phi = -1,$$
$$\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi = \frac{1}{r}\hat{D}_{\hat{Z}_A}(\Pi_{\tilde{Y}}), \qquad \nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi = -\frac{1}{r}(\tilde{\Pi}_{\hat{Z}}).$$

Lemma 5.3.5. *We have the following derivatives of $\frac{\beta(\cdot)}{2r}$:*

$$\mathscr{L}_R(\tilde{X}_A)|_q\frac{\beta(\cdot)}{2r} = -((\frac{\beta(q)}{2r})^2 + \tilde{\hat{\sigma}}^2_A), \qquad \mathscr{L}_{NS}(\tilde{Y}_A)|_q\frac{\beta(\cdot)}{2r} = 0,$$
$$\mathscr{L}_{NS}(A\tilde{X}_A)|_q\frac{\beta(\cdot)}{2r} = -((\frac{\beta(q)}{2r})^2 + \tilde{\hat{\sigma}}^2_A),$$
$$\nu(A(X \wedge Y))|_q\frac{\beta(\cdot)}{2r} = 0.$$

Lemma 5.3.6. *We have the following derivatives of $\tilde{\hat{\sigma}}^1_{(\cdot)}$:*

$$\mathscr{L}_{NS}(\tilde{X}_A)|_q\tilde{\hat{\sigma}}^1_{(\cdot)} = 0, \qquad \mathscr{L}_{NS}(\tilde{Y}_A)|_q\tilde{\hat{\sigma}}^1_{(\cdot)} = 0,$$
$$\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\tilde{\hat{\sigma}}^1_{(\cdot)} = -\frac{2}{r}\tilde{\Pi}_{\hat{Z}}(\tilde{\hat{\sigma}}^2_A - \hat{\sigma}_A), \qquad \nu(A(X \wedge Y))|_q\tilde{\hat{\sigma}}^1_{(\cdot)} = 0,$$
$$\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\tilde{\hat{\sigma}}^1_{(\cdot)} = +\frac{2}{r}(\tilde{\Pi}_{\hat{Z}})^2 - 2r.$$

Lemma 5.3.7. *We have the following derivatives of $\tilde{\hat{\sigma}}^2_{(\cdot)}$:*

$$\mathscr{L}_{NS}(\tilde{X}_A)|_q\tilde{\hat{\sigma}}^2_{(\cdot)} = 0, \qquad \mathscr{L}_{NS}(\tilde{Y}_A)|_q\tilde{\hat{\sigma}}^2_{(\cdot)} = 0,$$
$$\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\tilde{\hat{\sigma}}^2_{(\cdot)} = \frac{2}{r}\tilde{\Pi}_{\hat{Z}}(\tilde{\hat{\sigma}}^2_A - \hat{\sigma}_A), \qquad \nu(A(X \wedge Y))|_q\tilde{\hat{\sigma}}^2_{(\cdot)} = 0,$$
$$\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\tilde{\hat{\sigma}}^2_{(\cdot)} = -\frac{2}{r}(\tilde{\Pi}_{\hat{Z}})^2.$$

Lemma 5.3.8. *We have the following derivatives of $\tilde{\Pi}_{\hat{Z}}$:*

$$\mathscr{L}_{NS}(\tilde{X}_A)|_q\tilde{\Pi}_{\hat{Z}} = 0, \qquad \mathscr{L}_{NS}(\tilde{Y}_A)|_q\tilde{\Pi}_{\hat{Z}} = 0,$$
$$\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\tilde{\Pi}_{\hat{Z}} = \frac{1}{r}(\tilde{\hat{\sigma}}^1_A - \tilde{\hat{\sigma}}^2_A)(\tilde{\hat{\sigma}}^2_A - \hat{\sigma}_A) + r, \qquad \nu(A(X \wedge Y))|_q\tilde{\Pi}_{\hat{Z}} = 0,$$
$$\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\tilde{\Pi}_{\hat{Z}} = -\frac{1}{r}\tilde{\Pi}_{\hat{Z}}(\tilde{\hat{\sigma}}^1_A - \tilde{\hat{\sigma}}^2_A).$$

Proof. We will just prove fourth equalities from all the lemmas: $\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\hat{\sigma}_{(\cdot)}$, $\mathscr{L}_R(\tilde{X}_A)|_q r$, $\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi$ and $\mathscr{L}_R(\tilde{X}_A)|_q(\frac{\beta(\cdot)}{2r})$. Each one of the other derivatives can

be proved by using a method of the first fourth proofs.
The first derivative is

$$\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q \hat{\sigma}_{(\cdot)} = \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q \hat{g}(\hat{R}(\star\hat{Z}), \star\hat{Z}) = 2\hat{g}(\hat{R}(\star\tilde{Y}_A), \star\hat{Z}_A) = 2\Pi_{\tilde{Y}} = 0.$$

$\mathscr{L}_R(\tilde{X}_A)|_q r$ is deduced by using Second Bianchi Identity projected onto $\star\hat{Z}_A$ (see Equation (5.62)),

$$\begin{aligned}
\mathscr{L}_R(\tilde{X}_A)|_q r &= \mathscr{L}_R(\tilde{X}_A)|_q \Pi_{\tilde{X}} = \mathscr{L}_R(\tilde{X}_A)|_q (\hat{g}(\hat{R}(\star(\cdot)\tilde{X}), \star\hat{Z})) \\
&= \hat{g}((\hat{\nabla}_{A\tilde{X}_A}\hat{R})(\star A\tilde{X}_A), \star\hat{Z}_A) + \hat{g}(\hat{R}(\star A\mathscr{L}_R(\tilde{X}_A)|_q\tilde{X}), \star\hat{Z}_A) \\
&\quad + \hat{g}(\hat{R}(\star A\tilde{X}_A), \star\mathscr{L}_R(\tilde{X}_A)|_q\hat{Z}) \\
&= -\hat{g}((\hat{\nabla}_{\hat{Z}_A}\hat{R})(\star\hat{Z}_A), \star\hat{Z}_A) - \hat{g}((\hat{\nabla}_{A\tilde{Y}_A}\hat{R})(\star A\tilde{Y}_A), \star\hat{Z}_A) \\
&= -\beta(q) - \mathscr{L}_R(\tilde{Y}_A)|_q(\hat{g}(\hat{R}(\star(\cdot)\tilde{Y}), \star\hat{Z})) + \hat{g}(\hat{R}(\star A\mathscr{L}_R(\tilde{Y}_A)|_q\tilde{Y}), \star\hat{Z}_A) \\
&= -\beta(q) - \mathscr{L}_R(\tilde{Y}_A)|_q\Pi_{\tilde{Y}} - \tfrac{1}{2r}\beta(q)\Pi_{\tilde{X}},
\end{aligned}$$

so,

$$\mathscr{L}_R(\tilde{X}_A)|_q r = -\frac{3}{2}\beta(q).$$

We compute now the derivative of ϕ with respect to $\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q$ as follows,

$$\begin{aligned}
& -rs_\phi\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi + c_\phi\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q r = \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\Pi_X \\
= \; & -s_\phi\nu(\theta_X \otimes \hat{Z}_A)|_q\Pi_X + c_\phi\nu(\theta_Y \otimes \hat{Z}_A)|_q\Pi_X = -s_\phi(\hat{\sigma}_1 - \hat{\sigma}_A) - c_\phi\Pi_Z,
\end{aligned} \tag{5.17}$$

and,

$$\begin{aligned}
& rc_\phi\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi + s_\phi\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q r = \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\Pi_Y \\
= \; & -s_\phi\nu(\theta_X \otimes \hat{Z}_A)|_q\Pi_Y + c_\phi\nu(\theta_Y \otimes \hat{Z}_A)|_q\Pi_Y = s_\phi\Pi_Z + c_\phi(\hat{\sigma}_2 - \hat{\sigma}_A).
\end{aligned} \tag{5.18}$$

Thus, if we multiply (5.17) by $-s_\phi$ and (5.18) by c_ϕ and add the results then we get

$$\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi = \frac{1}{r}(s_\phi^2(\hat{\sigma}_1 - \hat{\sigma}_A) + 2c_\phi s_\phi\Pi_Z + c_\phi^2(\hat{\sigma}_2 - \hat{\sigma}_A)) = \frac{1}{r}(\tilde{\hat{\sigma}}_A^2 - \hat{\sigma}_A).$$

Finally, (5.2) and the integrability relation of the bracket $[\mathscr{L}_R(\tilde{X}), \mathscr{L}_R(\tilde{Y})]|_q$ allow us to obtain $\mathscr{L}_R(\tilde{X}_A)|_q(\frac{\beta(\cdot)}{2r})$. Indeed, we have

$$\begin{aligned}
[\mathscr{L}_R(\tilde{X}), \mathscr{L}_R(\tilde{Y})]|_q\phi = \; & \mathscr{L}_R(\tilde{X}_A)|_q(\mathscr{L}_R(\tilde{Y})|_q\phi) - \mathscr{L}_R(\tilde{Y}_A)|_q(\mathscr{L}_R(\tilde{X})|_q\phi) \\
= \; & -g(\nabla_{\tilde{X}_A}\Gamma, \tilde{Y}_A) + g(\nabla_{\tilde{Y}_A}\Gamma, \tilde{X}_A) + \tfrac{1}{2r}\beta(q)g(\Gamma, \tilde{Y}_A) + \mathscr{L}_R(\tilde{X}_A)|_q(\tfrac{1}{2r}\beta) \\
= \; & K(x) + \tfrac{1}{2r}\beta(q)g(\Gamma, \tilde{Y}_A) + \mathscr{L}_R(\tilde{X}_A)|_q(\tfrac{1}{2r}\beta),
\end{aligned}$$

while we also have,

$$\begin{aligned}
[\mathscr{L}_R(\tilde{X}),\mathscr{L}_R(\tilde{Y})]|_q\phi = &\ \mathscr{L}_R(\mathscr{L}_R(\tilde{X}_A)|_q\tilde{Y}-\mathscr{L}_R(\tilde{Y}_A)|_q\tilde{X})|_q\phi - r\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi \\
= &\ -\tfrac{1}{2r}\beta(q)\mathscr{L}_R(\tilde{Y}_A)|_q\phi - \tilde{\hat{\sigma}}_A^2+\hat{\sigma}_A \\
= &\ \tfrac{1}{2r}\beta(q)g(\Gamma,\tilde{Y}_A)-\tfrac{1}{4r^2}\beta^2(q)-\tilde{\hat{\sigma}}_A^2+\hat{\sigma}_A.
\end{aligned}$$

Then,

$$\mathscr{L}_R(\tilde{X}_A)|_q(\frac{\beta(\cdot)}{2r}) = -((\frac{\beta(q)}{2r})^2+\tilde{\hat{\sigma}}_A^2).$$

□

Remark 5.3.9. If we proceed to compute the tangent vectors fields on $\mathcal{O}_{\mathscr{D}_R}(q_0)$, we find that the first Lie bracket between $\mathscr{L}_R(\tilde{X}_A)|_q$ and $\mathscr{L}_R(\tilde{Y}_A)|_q$ and the assumptions of the current section, i.e. $\hat{\sigma}_A - K \equiv 0$ and $r\neq 0$ show us that $\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Despite that $[\mathscr{L}_R(\tilde{X}),\nu(\theta_{\tilde{Y}}\otimes\hat{Z}_A)]|_q$ didn't give us any new tangent vector, the bracket $[\mathscr{L}_R(\tilde{Y}),\nu(\theta_{\tilde{Y}}\otimes\hat{Z}_A)]|_q$ is equal to

$$\begin{aligned}
[\mathscr{L}_R(\tilde{Y}),\nu(\theta_{\tilde{Y}}\otimes\hat{Z}_A)]|_q =&(\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q - \frac{\beta(q)}{2r}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
&-\mathscr{L}_{NS}(\hat{Z}_A)|_q.
\end{aligned}$$

Then, there is a new tangent vector fields to $\mathcal{O}_{\mathscr{D}_R}(q_0)$ which is

$$F|_q := \mathscr{L}_{NS}(\hat{Z}_A)|_q + \frac{\beta(q)}{2r}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q. \tag{5.19}$$

Also, the Lie bracket between $\mathscr{L}_R(\tilde{Y}_A)|_q$ and $F|_q$ is equal to

$$\begin{aligned}
[\mathscr{L}_R(\tilde{Y}),F]|_q =&(F|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + (\tilde{\Pi}_Z+\mathscr{L}_R(\tilde{Y}_A)|_q(\frac{\beta(q)}{2r}))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
&- r\nu(A(X\wedge Y))|_q + (\tilde{\sigma}_A^1+(\frac{\beta(q)}{2r})^2)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q.
\end{aligned}$$

Using Lemma 5.3.2, the equality $0=[\mathscr{L}_R(\tilde{Y}),F]|_q\hat{\sigma}_{(\cdot)}-K$ implies that $(\tilde{\Pi}_Z+\mathscr{L}_R(\tilde{Y}_A)|_q(\frac{\beta(q)}{2r}))$ $\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\hat{\sigma}_{(\cdot)}=0$. However, $\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\hat{\sigma}_{(\cdot)}=r\neq 0$, then we have on $\mathcal{O}_{\mathscr{D}_R}(q_0)$

$$\mathscr{L}_R(\tilde{Y}_A)|_q(\frac{\beta(q)}{2r}) = -\tilde{\Pi}_Z.$$

Therefore, $[\mathscr{L}_R(\tilde{Y}),F]|_q$ is equal to

$$[\mathscr{L}_R(\tilde{Y}),F]|_q =(F|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q - r\nu(A(X\wedge Y))|_q + (\tilde{\sigma}_A^1+(\frac{\beta(q)}{2r})^2)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q.$$

Since $r\neq 0$, then $\nu(A(X\wedge Y))|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$.

By the computations of Lemma 5.3.1, the Lie brackets $[\mathscr{L}_R(\tilde{X}), \nu((\cdot)(X \wedge Y))]|_q$ and $[\mathscr{L}_R(\tilde{X}), \nu((\cdot)(X \wedge Y))]|_q$ imply that $\mathscr{L}_{NS}(A\tilde{X}_A)|_q$ and $\mathscr{L}_{NS}(A\tilde{Y}_A)|_q$ are also tangents to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Therefore, there are at least 7 vector fields tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$;

$$\mathscr{L}_R(\tilde{X}_A)|_q, \mathscr{L}_R(\tilde{Y}_A)|_q, \mathscr{L}_{NS}(A\tilde{X}_A)|_q, \mathscr{L}_{NS}(A\tilde{Y}_A)|_q, \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q, \nu(A(X \wedge Y))|_q, F|_q,$$

where $F|_q$ is the vector field given by (5.19).

Proposition 5.3.10. *If $K \equiv \hat{\sigma}_A$ on an open of $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ and $r \neq 0$ on the neighbourhood $\hat{V}$ of $\hat{x}$, then K is constant globally on M and β vanishes locally on $\hat{V}$.*

Proof. Using Remark 5.3.9, since $\mathscr{L}_R(\tilde{X}_A)|_q$, $\mathscr{L}_R(\tilde{Y}_A)|_q$, $\mathscr{L}_{NS}(A\tilde{X}_A)|_q$ and $\mathscr{L}_{NS}(A\tilde{Y}_A)|_q$ are tangents to $\mathcal{O}_{\mathscr{D}_R}(q_0)$ then so is also $\mathscr{L}_{NS}(\tilde{X}_A)|_q$ and $\mathscr{L}_{NS}(\tilde{Y}_A)|_q$. Moreover, $\mathscr{L}_{NS}(\tilde{X}_A)|_q\hat{\sigma}_{(\cdot)}$ $= \mathscr{L}_{NS}(\tilde{Y}_A)|_q\hat{\sigma}_{(\cdot)} \equiv 0$. Therefore, $\mathscr{L}_{NS}(\tilde{X}_A)|_q K = \mathscr{L}_{NS}(\tilde{Y}_A)|_q K \equiv 0$. This means that K is constant on M, and consequently, $\hat{\sigma}_A$ is constant on $\hat{V}$.
For $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, the matrix which represented $\hat{R}$ with respect to the orthonormal oriented basis $\star A\tilde{X}_A, \star A\tilde{Y}_A, \star\hat{Z}_A$ is,

$$\begin{pmatrix} -\tilde{\hat{\sigma}}_A^1 & \tilde{\Pi}_{\hat{Z}} & r \\ \tilde{\Pi}_{\hat{Z}} & -\tilde{\hat{\sigma}}_A^2 & 0 \\ r & 0 & -K \end{pmatrix}.$$

We know that the eigenvalues of $\hat{R}$ depend only of the fixed point $\hat{x}$, this implies that the characteristic polynomial $f(\tau)$ of $\hat{R}$ depends only of $\hat{x}$. Therefore, the derivatives of the coefficients of f with respect to any vector field of $T\mathcal{O}_{\mathscr{D}_R}(q_0)$ tangent to M vanishes. f is equal to

$$\begin{aligned} f(\tau) = &-\tau^3 + (-K - \tilde{\hat{\sigma}}_A^1 - \tilde{\hat{\sigma}}_A^2)\tau^2 + (r^2 - K\tilde{\hat{\sigma}}_A^1 - K\tilde{\hat{\sigma}}_A^2 - \tilde{\hat{\sigma}}_A^1\tilde{\hat{\sigma}}_A^2 + (\tilde{\Pi}_{\hat{Z}})^2)\tau \\ &+ (r^2\tilde{\hat{\sigma}}_A^2 - K\tilde{\hat{\sigma}}_A^1\tilde{\hat{\sigma}}_A^2 + K(\tilde{\Pi}_{\hat{Z}})^2). \end{aligned}$$

Referring to the derivatives of $\hat{\sigma}_A$, r and $\tilde{\hat{\sigma}}_A^2$ along $\mathscr{L}_{NS}(\tilde{X}_A)|_q$ in Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.7 respectively, the derivation of the third coefficient of f is,

$$\mathscr{L}_{NS}(\tilde{X}_A)|_q(r^2 - K\tilde{\hat{\sigma}}_{(\cdot)}^1 - K\tilde{\hat{\sigma}}_{(\cdot)}^2 - \tilde{\hat{\sigma}}_{(\cdot)}^1\tilde{\hat{\sigma}}_{(\cdot)}^2 + (\tilde{\Pi}_{\hat{Z}})^2) = \mathscr{L}_{NS}(\tilde{X}_A)|_q(r^2) = -\beta(q)r.$$

As we said before, the previous derivation is equal to zero. Thus,

$$\beta(q) = 0.$$

□

Proposition 5.3.11. *Under the same assumptions of Proposition 5.3.10, we have*

a) $\tilde{\hat{\sigma}}_A^2$ and $\tilde{\Pi}_{\hat{Z}_A}$ vanish locally on $\mathcal{O}_{\mathscr{D}_R}(q_0)$,

b) We have $r^2 = K\tilde{\hat{\sigma}}_A^1$, in particular $K \neq 0$ and $\tilde{\hat{\sigma}}_A^1 \neq 0$.

Proof.

a) Since $\beta(q)=0$ then $F|_q=\mathscr{L}_{NS}(\hat{Z}_A)|_q$ is a tangent vector to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. The Lie brackets $[\mathscr{L}_R(\tilde{X}),\mathscr{L}_{NS}(\hat{Z})]|_q$ and $[\mathscr{L}_R(\tilde{Y}),\mathscr{L}_{NS}(\hat{Z})]|_q$ mentioned in Lemma 5.3.1 are

$$\begin{aligned}&[\mathscr{L}_R(\tilde{X}),\mathscr{L}_{NS}(\hat{Z})]|_q\\ =&-(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q-\tilde{\hat{\sigma}}_A^2\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q+\tilde{\Pi}_{\hat{Z}_A}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,\end{aligned}$$

$$\begin{aligned}&[\mathscr{L}_R(\tilde{Y}),\mathscr{L}_{NS}(\hat{Z})]|_q\\ =&(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q+\tilde{\Pi}_{\hat{Z}_A}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q+\tilde{\hat{\sigma}}_A^1\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q-r\nu(A(X\wedge Y))|_q.\end{aligned}$$

By Lemma 5.3.2, the derivatives of $\hat{\sigma}_{(\cdot)}$ along these brackets are

$$\begin{aligned}[\mathscr{L}_R(\tilde{X}),\mathscr{L}_{NS}(\hat{Z})]|_q\hat{\sigma}_{(\cdot)}=&-(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q\hat{\sigma}_{(\cdot)}-2\tilde{\hat{\sigma}}_A^2r,\\ [\mathscr{L}_R(\tilde{Y}),\mathscr{L}_{NS}(\hat{Z})]|_q\hat{\sigma}_{(\cdot)}=&(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q\hat{\sigma}_{(\cdot)}+2\tilde{\Pi}_{\hat{Z}_A}r.\end{aligned}$$

On the other hand, the derivatives of K along them are

$$\begin{aligned}[\mathscr{L}_R(\tilde{X}),\mathscr{L}_{NS}(\hat{Z})]|_q\hat{\sigma}_{(\cdot)}=&-(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_qK,\\ [\mathscr{L}_R(\tilde{Y}),\mathscr{L}_{NS}(\hat{Z})]|_q\hat{\sigma}_{(\cdot)}=&(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_qK.\end{aligned}$$

Since $K-\hat{\sigma}_A\equiv 0$ and $r\neq 0$, then

$$\tilde{\hat{\sigma}}_A^2\equiv 0,\quad \tilde{\Pi}_{\hat{Z}_A}\equiv 0.$$

b) Since $\tilde{\Pi}_{\hat{Z}_A}=0$ and $\tilde{\hat{\sigma}}_A^2=0$, we can compute

$$0=\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\tilde{\Pi}_{\hat{Z}}=\frac{1}{r}(-\hat{\sigma}_A\tilde{\hat{\sigma}}_A^1+r^2).$$

So, $\hat{\sigma}_A\tilde{\hat{\sigma}}_A^1=r^2$, hence $\tilde{\hat{\sigma}}_A^1\neq 0$ and $K=\hat{\sigma}_A\neq 0$.

□

Proposition 5.3.12. *Under the same assumptions of Proposition 5.3.10, the dimension of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is 7 and it is generated by*

$$\mathscr{L}_R(\tilde{X}_A)|_q,\mathscr{L}_R(\tilde{Y}_A)|_q,\mathscr{L}_{NS}(A\tilde{X}_A)|_q,\mathscr{L}_{NS}(A\tilde{Y}_A)|_q,\mathscr{L}_{NS}(\hat{Z}_A)|_q,\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,\nu(A(X\wedge Y))|_q. \tag{5.20}$$

Proof. From Remark5.3.9, we know that

$\mathscr{L}_R(\tilde{X}_A)|_q,\mathscr{L}_R(\tilde{Y}_A)|_q,\mathscr{L}_{NS}(A\tilde{X}_A)|_q,\mathscr{L}_{NS}(A\tilde{Y}_A)|_q,\mathscr{L}_{NS}(\hat{Z}_A)|_q,\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q$ and $\nu(A(X\wedge Y))|_q$

are tangent vector fields to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. This means that the distribution generated by these vectors fields is included in the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Reciprocally, we start by searching the value of $\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi$ because we have

$$[\nu(\theta_{\tilde{Y}} \otimes \hat{Z}), \mathscr{L}_{NS}(\hat{Z})]|_q = (\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q - \mathscr{L}_{NS}(A\tilde{Y}_A)|_q.$$

Using Proposition 5.3.11 and the fact that we have $\mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi = \mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi = 0$ by Propositions 5.3.10, to compute

$$\begin{aligned}\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi =& \frac{1}{r}\hat{D}_{\hat{Z}_A}(\tilde{\Pi}_{\tilde{Y}}) = \frac{1}{r}\hat{g}((\hat{\nabla}_{\hat{Z}_A}\hat{R})(\star\hat{Z}_A), \star A\tilde{Y}_A) \\ =& \frac{1}{r}(-\hat{g}((\hat{\nabla}_{A\tilde{X}_A}\hat{R})(\star A\tilde{X}_A), \star A\tilde{Y}_A) - \hat{g}((\hat{\nabla}_{A\tilde{Y}_A}\hat{R})(\star A\tilde{Y}_A), \star A\tilde{Y}_A)) \\ =& \frac{1}{r}(-\mathscr{L}_{NS}(A\tilde{X}_A)|_q\tilde{\Pi}_{\hat{Z}} - \mathscr{L}_{NS}(A\tilde{Y}_A)|_q\tilde{\hat{\sigma}}^2_{(\cdot)}) = \frac{1}{r}(0) = 0.\end{aligned}$$

By this and Lemma 5.3.1, the distribution generated by the seven vectors fields in (5.21) is involutive which proves the claim and finishes the proof. ☐

Corollary 5.3.13. *Under the same assumptions of Proposition 5.3.10, $\hat{R}$ has 0 as a double eigenvalue and $-K - \tilde{\hat{\sigma}}^1_A$ as a simple eigenvalue.*

Proof. By Proposition 5.3.11, the matrix represented $\hat{R}$ has the form

$$\begin{pmatrix} -\tilde{\hat{\sigma}}^1_A & 0 & r \\ 0 & 0 & 0 \\ r & 0 & -K \end{pmatrix}.$$

So, using $K\tilde{\hat{\sigma}}^1_A = r^2$, 0 is a double eigenvalue of $\hat{R}$ with the corresponding eigenvectors $\star A\tilde{Y}_A$ and $\star W_A := K \star A\tilde{X}_A + r \star \hat{Z}_A$. Moreover, $\hat{R}$ cannot be a null symmetric matrix, then the third eigenvalue $-K_A$ is necessarily nonzero and it is equal to the trace of $\hat{R}$, i.e. $-K_A = -K - \tilde{\hat{\sigma}}^1_A$. Its eigenvector is $\star V_A := -r \star A\tilde{X}_A + K \star \hat{Z}_A$. Also, the equation $K\tilde{\hat{\sigma}}^1_A = r^2$ implies that $\tilde{\hat{\sigma}}^1_A$, K and K_A have same sign. ☐

Corollary 5.3.14. *For every $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, on the neighbourhood $\hat{V}$ of $\hat{x}$, there is an oriented orthonormal frame $\hat{E}_1, \hat{E}_2, \hat{E}_3$ on $\hat{M}$ with respect to which the connection table are of the form*

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)} \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.21}$$

Proof. For $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, let X, Y be a locally orthonormal oriented frame of (M, g) and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ be an orthonormal frame of $(\hat{M}, \hat{g})$. We denote by $-\hat{K}_1, -\hat{K}_2, -\hat{K}_3$ the eigenvalues of $\hat{R}$ corresponding to $\star\hat{E}_1, \star\hat{E}_2, \star\hat{E}_3$. With the fact that $\hat{R}$ is an nonzero matrix, then there is an nonzero eigenvalue which we take it

$-\hat{K}_2$. By Corollary 5.3.13, $-K_A$ is the simple nonzero eigenvalue of $\hat{R}$. Thus $\hat{E}_2 = \pm\tilde{V}_A := \frac{V_A}{\|V_A\|_{\hat{g}}}$. However, $\nu(A(X\wedge Y))|_q$ and $\nu(\theta_{\tilde{Y}_A}\otimes \hat{Z}_A)|_q$ are tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$ then $\nu(\star\hat{E}_2A)|_q = \pm\nu(\star\tilde{V}_AA)|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Since $\tilde{V}_A^{\perp} = span\{\tilde{W}_A := \frac{W_A}{\|W_A\|_{\hat{g}}}, A\tilde{Y}_A\}$ and $\hat{E}_2^{\perp} = span\{\hat{E}_1,\hat{E}_3\}$ then one may define the smooth function $\zeta : \mathcal{O}_{\mathscr{D}_R}(q_0)\to\mathbb{R}$ such that,

$$\begin{aligned}\tilde{W}_A &= cos(\zeta(q))\hat{E}_1 + sin(\zeta(q))\hat{E}_3,\\ A\tilde{Y}_A &= -sin(\zeta(q))\hat{E}_1 + cos(\zeta(q))\hat{E}_3.\end{aligned}$$

In order to find the table form (5.21), we begin by computing in $\mathcal{O}_{\mathscr{D}_R}(q_0)$,

$$[\mathscr{L}_{NS}(\hat{E}_2),\nu(\star\hat{E}_2(\cdot))]|_q = \nu(\star(-\hat{\Gamma}^2_{(1,2)}\hat{E}_1 + \hat{\Gamma}^2_{(2,3)}\hat{E}_3)A)|_q := F_1|_q.$$

We get that F_1 is a vector field on $\mathcal{O}_{\mathscr{D}_R}(q_0)$, then we also able to compute

$$[F_1,\nu(\star\hat{E}_2(\cdot))]|_q = \nu(\star(-\hat{\Gamma}^2_{(1,2)}\hat{E}_3 - \hat{\Gamma}^2_{(2,3)}\hat{E}_1)A)|_q := F_2|_q,$$

and hence F_2 is a vector field of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ as well. If $\nu(\star\hat{E}_2A)|_q, F_1|_q, F_2|_q$ are linearly independent for $q\in\mathcal{O}_{\mathscr{D}_R}(q_0)$, it would follow that they form a basis of $V|_q(\pi_Q)$ for $q\in\mathcal{O}_{\mathscr{D}_R}(q_0)$ and hence $V|_q(\pi_Q)\subset T_q(\mathcal{O}_{\mathscr{D}_R}(q_0))$ for $q\in\mathcal{O}_{\mathscr{D}_R}(q_0)$. Then Corollary 5.6 in [28] would imply that $(\Sigma)_R$ is completely controllable and $\mathcal{O}_{\mathscr{D}_R}(q_0)$ will be of dimension 8, which is a contradiction. Hence, in a dense subset O of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ one has that $\nu(\star\hat{E}_2A)|_q, F_1|_q, F_2|_q$ are linearly dependent. This implies that

$$0 = det\begin{pmatrix} 0 & 1 & 0\\ -\hat{\Gamma}^2_{(1,2)} & 0 & \hat{\Gamma}^2_{(2,3)}\\ -\hat{\Gamma}^2_{(2,3)} & 0 & -\hat{\Gamma}^2_{(1,2)}\end{pmatrix} = -((\hat{\Gamma}^2_{(1,2)})^2 + (\hat{\Gamma}^2_{(2,3)})^2)$$

Therefore, $\hat{\Gamma}^2_{(1,2)} = 0$ and $\hat{\Gamma}^2_{(2,3)} = 0$ on $\pi_{\mathcal{O}_{\mathscr{D}_R}(q_0),M}(O)$. Repeat this method with taking firstly the bracket $[\mathscr{L}_{NS}(\hat{E}_1),\nu(\star\hat{E}_2(\cdot))]|_q$ and then with $[\mathscr{L}_{NS}(\hat{E}_3),\nu(\star\hat{E}_2(\cdot))]|_q$, we, respectively, obtain that

$$\begin{aligned}\hat{\Gamma}^1_{(1,2)} = 0 \ and\ \hat{\Gamma}^1_{(2,3)} = 0,\\ \hat{\Gamma}^3_{(1,2)} = 0 \ and\ \hat{\Gamma}^3_{(2,3)} = 0.\end{aligned}$$

□

Theorem 5.3.15. *For every $q = (x,\hat{x};A)\in\mathcal{O}_{\mathscr{D}_R}(q_0)$ there is an open neighbourhood of $\hat{V}\ni\hat{x}$ such that $(\hat{V},\hat{g})$ is isometry to the Riemannian product $(I\times\hat{N},\hat{s})$, where $I\subset\mathbb{R}$ is an open interval and N is a 2-dimensional Riemannian manifold.*

Proof. Remark that the results of Corollary 5.3.14 imply that the assumptions of Theorem 5.4.4 are fulfilled. Thus, there is a neighbourhood, which we also denote it by $\hat{V}$, of $\hat{x}$, an interval $I\subset\mathbb{R}$, $\hat{f}\in C^\infty(I)$ and a 2-dimensional Riemannian manifold

$(\hat{N},\hat{h})$ such that $(\hat{V},\hat{g}|_{\hat{V}})$ is isometric to the warped product $(I\times N,h_{\hat{f}})$ where $h_{\hat{f}}:=pr_1^*(\hat{g})+(\hat{f}\circ pr_1)^2pr_2^*(\hat{h}))$ and such that

$$\frac{\hat{f}'(r)}{\hat{f}(r)}=-\hat{\Gamma}^1_{(1,2)}(\hat{x})=0.$$

Therefore, $\hat{f}$ is a constant function i.e. $(I\times N,h_{\hat{f}})$ is a Riemannian product. □

5.3.2 Case $(\Pi_X,\Pi_Y)\neq(0,0)$ and $\hat{\sigma}_A\neq K(x)$

Here, we suppose that $\hat{\sigma}_A\neq K(x)$ on open set of $q=(x,\hat{x};A)\in\mathcal{O}_{\mathscr{D}_R}(q_0)$ and $(\Pi_X,\Pi_Y)\neq(0,0)$ on the neighbourhood $\hat{V}$ of $\hat{x}$. The rolling curvature tensor Rol is equal to

$$\mathsf{Rol}_q(X,Y)=-(K-\hat{\sigma}_A)\Big(A(X\wedge Y)-\frac{\Pi_Y}{K-\hat{\sigma}_A}\theta_X\otimes\hat{Z}_A+\frac{\Pi_X}{K-\hat{\sigma}_A}\theta_Y\otimes\hat{Z}_A\Big).$$

Since $K-\hat{\sigma}_A\neq 0$ on $\mathcal{O}_{\mathscr{D}_R}(q_0)$, we can define $\omega:=\frac{r}{K-\hat{\sigma}_A}$ and hence

$$\begin{cases}\frac{\Pi_X}{K-\hat{\sigma}_A} &=\omega c_\phi,\\ \frac{\Pi_Y}{K-\hat{\sigma}_A} &=\omega s_\phi,\end{cases}\qquad\qquad \begin{cases}\Pi_{\tilde{X}} &=\omega(K-\hat{\sigma}_A),\\ \Pi_{\tilde{Y}} &=0.\end{cases}$$

Using (5.10), Rol is equal to

$$\begin{aligned}\mathsf{Rol}_q(X,Y)&=-(K-\hat{\sigma}_A)\big(A(X\wedge Y)+\omega\theta_{\tilde{Y}_A}\otimes\hat{Z}_A\big)\\ &=:-(K-\hat{\sigma}_A)\overline{\mathsf{Rol}}_q(X,Y),\end{aligned}$$

Lemma 5.3.16. *We have the following formulas on $\mathcal{O}_{\mathscr{D}_R}(q_0)$:*

$$\nu(A(X\wedge Y))|_q\phi=-1,\tag{5.22}$$
$$\nu(A(X\wedge Y))|_q\omega=0,\tag{5.23}$$
$$\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi=\frac{1}{\omega(K-\hat{\sigma}_A)}(\tilde{\hat{\sigma}}^2_A-\hat{\sigma}_A),\tag{5.24}$$
$$\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\omega=\frac{-\tilde{\Pi}_{\hat{Z}}}{K-\hat{\sigma}_A},\tag{5.25}$$
$$\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\phi=\frac{-\tilde{\Pi}_{\hat{Z}}}{\omega(K-\hat{\sigma}_A)},\tag{5.26}$$
$$\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\omega=\frac{1}{K-\hat{\sigma}_A}(\tilde{\hat{\sigma}}^1_A-\hat{\sigma}_A)+2\omega^2,\tag{5.27}$$
$$\mathscr{L}_{NS}(\tilde{X}_A)|_q\phi=-g(\Gamma,\tilde{X}_A),\tag{5.28}$$
$$\mathscr{L}_{NS}(\tilde{Y}_A)|_q\phi=-g(\Gamma,\tilde{Y}_A).\tag{5.29}$$

Proof. We extract by the same method every two derivatives of ϕ and ω with respect to the same vector fields. We will explicit this for the first two formulas, i.e. (5.22) and (5.23), such we have

$$\begin{aligned}\nu(A(X\wedge Y))|_q\omega c_\phi &= -s_\phi\omega\nu(A(X\wedge Y))|_q\phi + c_\phi\nu(A(X\wedge Y))|_q\omega,\\ \nu(A(X\wedge Y))|_q\omega s_\phi &= c_\phi\omega\nu(A(X\wedge Y))|_q\phi + s_\phi\nu(A(X\wedge Y))|_q\omega.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\nu(A(X\wedge Y))|_q\omega c_\phi &= \nu(A(X\wedge Y))|_q\frac{\Pi_X}{K-\hat{\sigma}_{(\cdot)}} = \frac{\Pi_Y}{K-\hat{\sigma}_A} = \omega s_\phi,\\ \nu(A(X\wedge Y))|_q\omega s_\phi &= \nu(A(X\wedge Y))|_q\frac{\Pi_Y}{K-\hat{\sigma}_{(\cdot)}} = \frac{-\Pi_X}{K-\hat{\sigma}_A} = -\omega c_\phi.\end{aligned}$$

By comparison, we obtain

$$\begin{cases} -s_\phi\omega\nu(A(X\wedge Y))|_q\phi + c_\phi\nu(A(X\wedge Y))|_q\omega &= \omega s_\phi,\\ c_\phi\omega\nu(A(X\wedge Y))|_q\phi + s_\phi\nu(A(X\wedge Y))|_q\omega &= -\omega c_\phi.\end{cases}$$

Multiplying the first equality of the previous system by c_ϕ and the second one by s_ϕ and adding the results, then we get

$$\nu(A(X\wedge Y))|_q\omega = 0.$$

Change the role of the multiplication of c_ϕ and s_ϕ by the equations of the previous and add the results. Since $\omega\neq 0$, then

$$\nu(A(X\wedge Y))|_q\phi = -1.$$

□

Remark 5.3.17. The first order of the tangent Lie brackets to $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is

$$\begin{aligned}[\mathscr{L}_R(\tilde{X}),\mathscr{L}_R(\tilde{Y})]|_q =& (-\mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + (-\mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q\\ &- (K-\hat{\sigma}_A)\nu(\overline{\mathsf{Rol}_q})|_q.\end{aligned}$$

They second order are

$$\begin{aligned}&[\mathscr{L}_R(\tilde{X}),\nu(\overline{\mathsf{Rol}_{(\cdot)}})]|_q\\ =&[\mathscr{L}_R(\tilde{X}),\nu(A(X\wedge Y))]|_q + (\mathscr{L}_R(\tilde{X}_A)|_q\omega)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q + \omega[\mathscr{L}_R(\tilde{X}),\nu(\theta_{\tilde{Y}}\otimes\hat{Z})]|_q\\ =&(1-\omega\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q - \mathscr{L}_{NS}(A\tilde{Y}_A)|_q\\ &+\omega(-\mathscr{L}_R(\tilde{X}_A)|_q\phi - g(\Gamma,\tilde{X}_A))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q + (\mathscr{L}_R(\tilde{X}_A)|_q\omega)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,\end{aligned}$$

$$\begin{aligned}&[\mathscr{L}_R(\tilde{Y}),\nu(\overline{\mathsf{Rol}_{(\cdot)}})]|_q\\ =&[\mathscr{L}_R(\tilde{Y}),\nu(A(X\wedge Y))]|_q + (\mathscr{L}_R(\tilde{Y}_A)|_q\omega)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q + \omega[\mathscr{L}_R(\tilde{Y}),\nu(\theta_{\tilde{Y}}\otimes\hat{Z})]|_q\\ =&(-1+\omega\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + \mathscr{L}_{NS}(A\tilde{X}_A)|_q - \omega\mathscr{L}_{NS}(\hat{Z}_A)|_q\\ &+\omega(-\mathscr{L}_R(\tilde{Y}_A)|_q\phi - g(\Gamma,\tilde{Y}_A))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q + (\mathscr{L}_R(\tilde{Y}_A)|_q\omega)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q.\end{aligned}$$

In order to simplify the notations, we propose the following terms

$$\begin{aligned}
G_{\tilde{X}} &:= \mathscr{L}_{NS}(A\tilde{X}_A)|_q\phi = \mathscr{L}_R(\tilde{X}_A)|_q\phi + g(\Gamma,\tilde{X}_A),\\
G_{\tilde{Y}} &:= \mathscr{L}_{NS}(A\tilde{Y}_A)|_q\phi = \mathscr{L}_R(\tilde{Y}_A)|_q\phi + g(\Gamma,\tilde{Y}_A),\\
H_{\tilde{X}} &:= \mathscr{L}_R(\tilde{X}_A)|_q\omega,\\
H_{\tilde{Y}} &:= \mathscr{L}_R(\tilde{Y}_A)|_q\omega.
\end{aligned}$$

By Lemma 5.3.1, remark that

$$[\mathscr{L}_{NS}((\cdot)\tilde{X}),\nu((\cdot)(X\wedge Y))]|_q = [\mathscr{L}_{NS}((\cdot)\tilde{Y}),\nu((\cdot)(X\wedge Y))]|_q = 0.$$

The integrability relation of each of the two previous brackets with respect to ϕ allows us to have,

$$\nu(A(X\wedge Y))|_q(G_{\tilde{X}}) = \nu(A(X\wedge Y))|_q(G_{\tilde{Y}}) = 0. \tag{5.30}$$

Remark 5.3.18. From the above remark, the two new tangent vector fields appeared are

$$\begin{aligned}
F_1|_q :=& -\mathscr{L}_{NS}(A\tilde{Y}_A)|_q - \omega G_{\tilde{X}}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q + H_{\tilde{X}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,\\
F_2|_q :=& \mathscr{L}_{NS}(A\tilde{X}_A)|_q - \omega\mathscr{L}_{NS}(\hat{Z}_A)|_q - \omega G_{\tilde{Y}}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q + H_{\tilde{Y}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q.
\end{aligned}$$

The third order tangent Lie brackets to $\mathcal{O}_{\mathscr{D}_R}(q_0)$

$$\begin{aligned}
& [\mathscr{L}_R(\tilde{X}),F_1]|_q\\
=\ & -(F_1|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q + G_{\tilde{X}}\mathscr{L}_{NS}(A\tilde{X}_A)|_q + \omega G_{\tilde{X}}\mathscr{L}_{NS}(\hat{Z}_A)|_q - \hat{\sigma}_A\nu(\overline{\mathsf{Rol}}_q)|_q\\
& -(2H_{\tilde{X}}G_{\tilde{X}} + \omega\mathscr{L}_R(\tilde{X}_A)|_q(G_{\tilde{X}}))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\\
& +(K\omega - \omega(G_{\tilde{X}})^2 + \mathscr{L}_R(\tilde{X}_A)|_q(H_{\tilde{X}}))\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}),F_1]|_q\\
=\ & (F_1|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + G_{\tilde{Y}}\mathscr{L}_{NS}(A\tilde{X}_A)|_q - H_{\tilde{X}}\mathscr{L}_{NS}(\hat{Z}_A)|_q\\
& -(G_{\tilde{X}}H_{\tilde{Y}} + G_{\tilde{Y}}H_{\tilde{X}} + \omega\mathscr{L}_R(\tilde{Y}_A)|_q(G_{\tilde{X}}))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\\
& -(\omega G_{\tilde{X}}G_{\tilde{Y}} - \mathscr{L}_R(\tilde{Y}_A)|_q(H_{\tilde{X}}))\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q
\end{aligned}$$

$$\begin{aligned}
& [\nu(\overline{\mathsf{Rol}}_{(\cdot)}),F_1]|_q\\
=\ & \omega(\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{X}_A)|_q - \omega\mathscr{L}_{NS}(\hat{Z}_A)|_q + \omega^2 G_{\tilde{X}}\nu(\overline{\mathsf{Rol}}_q)|_q\\
& -\omega(-F_1|_q\phi + H_{\tilde{X}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi + G_{\tilde{X}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\omega + \nu(\overline{\mathsf{Rol}}_q)|_q(G_{\tilde{X}}))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\\
& -(\omega^3 G_{\tilde{X}} + F_1|_q\omega + \omega^2 G_{\tilde{X}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi - \nu(\overline{\mathsf{Rol}}_q)|_q(H_{\tilde{X}}))\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$\begin{aligned}
& [\mathscr{L}_R(\tilde{X}),F_2]|_q\\
=\ & -(F_2|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q + G_{\tilde{X}}\mathscr{L}_{NS}(A\tilde{Y}_A)|_q + (\omega G_{\tilde{Y}} - H_{\tilde{X}})\mathscr{L}_{NS}(\hat{Z}_A)|_q\\
& -(H_{\tilde{X}}G_{\tilde{Y}} + H_{\tilde{Y}}G_{\tilde{X}} + \omega\hat{\tilde{\sigma}}_A^2 + \omega\mathscr{L}_R(\tilde{X}_A)|_q(G_{\tilde{Y}}))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\\
& -(\omega G_{\tilde{X}}G_{\tilde{Y}} + \omega\tilde{\Pi}_{\hat{Z}} - \mathscr{L}_R(\tilde{X}_A)|_q(H_{\tilde{Y}}))\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), F_2]|_q \\
= \ & (F_2|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + G_{\tilde{Y}}\mathscr{L}_{NS}(A\tilde{Y}_A)|_q - 2H_{\tilde{Y}}\mathscr{L}_{NS}(\hat{Z}_A)|_q - (\hat{\sigma}_A - (K-\hat{\sigma}_A)\omega^2)\nu(\overline{\mathsf{Rol}}_q)|_q \\
& -(2G_{\tilde{Y}}H_{\tilde{Y}} + \omega\tilde{\Pi}_{\hat{Z}} + \omega\mathscr{L}_R(\tilde{Y}_A)|_q(G_{\tilde{Y}}))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
& -(\omega(G_{\tilde{Y}})^2 + \omega(\tilde{\hat{\sigma}}_A^1 - K) + \omega^3(K-\hat{\sigma}_A) + \mathscr{L}_R(\tilde{Y}_A)|_q(H_{\tilde{Y}}))\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\nu(\overline{\mathsf{Rol}}_{(\cdot)}), F_2]|_q \\
= \ & (\omega^2 + \omega\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_{NS}(A\tilde{Y}_A)|_q - \omega(\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\omega)\mathscr{L}_{NS}(\hat{Z}_A)|_q + \omega^2 G_{\tilde{Y}}\nu(\overline{\mathsf{Rol}}_q)|_q \\
& -\omega(-F_2|_q\phi + H_{\tilde{Y}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi + G_{\tilde{Y}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\omega + \nu(\overline{\mathsf{Rol}}_q)|_q(G_{\tilde{Y}}))\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
& +(-\omega^3 G_{\tilde{Y}} - F_2|_q\omega - \omega^2 G_{\tilde{Y}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi + \nu(\overline{\mathsf{Rol}}_q)|_q(H_{\tilde{Y}}))\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q.
\end{aligned}
$$

If we try to write the previous vector fields in form of vectors of length 5 with respect to the vectors fields $\mathscr{L}_{NS}(A\tilde{X}_A)|_q, \mathscr{L}_{NS}(A\tilde{Y}_A)|_q, \mathscr{L}_{NS}(\hat{Z}_A)|_q, \nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q, \nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q$, then we get that

$$
F_1|_q = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -\omega G_{\tilde{X}} \\ H_{\tilde{X}} \end{pmatrix},
$$

$$
F_2|_q = \begin{pmatrix} 1 \\ 0 \\ -\omega \\ -\omega G_{\tilde{Y}} \\ H_{\tilde{Y}} \end{pmatrix},
$$

$$
[\mathscr{L}_R(\tilde{X}), F_1]|_q + (F_1|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q + \hat{\sigma}_A\nu(\overline{\mathsf{Rol}}_q)|_q = \begin{pmatrix} G_{\tilde{X}} \\ 0 \\ \omega G_{\tilde{X}} \\ -2H_{\tilde{X}}G_{\tilde{X}} - \omega\mathscr{L}_R(\tilde{X}_A)|_q(G_{\tilde{X}}) \\ K\omega - \omega(G_{\tilde{X}})^2 + \mathscr{L}_R(\tilde{X}_A)|_q(H_{\tilde{X}}) \end{pmatrix},
$$

$$
[\mathscr{L}_R(\tilde{Y}), F_1]|_q - (F_1|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q = \begin{pmatrix} G_{\tilde{Y}} \\ 0 \\ -H_{\tilde{X}} \\ -G_{\tilde{X}}H_{\tilde{Y}} - G_{\tilde{Y}}H_{\tilde{X}} - \omega\mathscr{L}_R(\tilde{Y}_A)|_q(G_{\tilde{X}}) \\ -\omega G_{\tilde{X}}G_{\tilde{Y}} + \mathscr{L}_R(\tilde{Y}_A)|_q(H_{\tilde{X}}) \end{pmatrix},
$$

$$[\mathscr{L}_R(\tilde{X}), F_2]|_q + (F_2|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q = \begin{pmatrix} 0 \\ G_{\tilde{X}} \\ \omega G_{\tilde{Y}} - H_{\tilde{X}} \\ -H_{\tilde{X}} G_{\tilde{Y}} - H_{\tilde{Y}} G_{\tilde{X}} - \omega \tilde{\hat{\sigma}}_A^2 - \omega \mathscr{L}_R(\tilde{X}_A)|_q(G_{\tilde{Y}}) \\ -\omega G_{\tilde{X}} G_{\tilde{Y}} - \omega \tilde{\Pi}_{\hat{Z}} + \mathscr{L}_R(\tilde{X}_A)|_q(H_{\tilde{Y}}) \end{pmatrix},$$

$$[\mathscr{L}_R(\tilde{Y}), F_2]|_q - (F_2|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q + (\hat{\sigma}_A - (K - \hat{\sigma}_A)\omega^2)\nu(\overline{\mathsf{Rol}_q})|_q$$

$$= \begin{pmatrix} 0 \\ G_{\tilde{Y}} \\ -2H_{\tilde{Y}} \\ -2G_{\tilde{Y}} H_{\tilde{Y}} - \omega \tilde{\Pi}_{\hat{Z}} - \omega \mathscr{L}_R(\tilde{Y}_A)|_q(G_{\tilde{Y}}) \\ -\omega (G_{\tilde{Y}})^2 - \omega(\tilde{\hat{\sigma}}_A^1 - K) - \omega^3(K - \hat{\sigma}_A) - \mathscr{L}_R(\tilde{Y}_A)|_q(H_{\tilde{Y}}) \end{pmatrix},$$

$$[\nu(\overline{\mathsf{Rol}_{(\cdot)}}), F_1]|_q - (\omega^2 G_{\tilde{X}})\nu(\overline{\mathsf{Rol}_q})|_q$$

$$= \begin{pmatrix} \omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi \\ 0 \\ -\omega \\ \omega(F_1|_q\phi - H_{\tilde{X}}\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi - G_{\tilde{X}}\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\omega - \nu(\overline{\mathsf{Rol}_q})|_q(G_{\tilde{X}})) \\ -\omega^3 G_{\tilde{X}} - F_1|_q\omega - \omega^2 G_{\tilde{X}}\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi + \nu(\overline{\mathsf{Rol}_q})|_q(H_{\tilde{X}}) \end{pmatrix},$$

$$[\nu(\overline{\mathsf{Rol}_{(\cdot)}}), F_2]|_q - (\omega^2 G_{\tilde{Y}})\nu(\overline{\mathsf{Rol}_q})|_q$$

$$= \begin{pmatrix} 0 \\ \omega^2 + \omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi \\ -\omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\omega \\ \omega(F_2|_q\phi - H_{\tilde{Y}}\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi - G_{\tilde{Y}}\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\omega - \nu(\overline{\mathsf{Rol}_q})|_q(G_{\tilde{Y}})) \\ -\omega^3 G_{\tilde{Y}} - F_2|_q\omega - \omega^2 G_{\tilde{Y}}\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi + \nu(\overline{\mathsf{Rol}_q})|_q(H_{\tilde{Y}}) \end{pmatrix}.$$

We conclude from the coefficients of $\mathscr{L}_{NS}(A\tilde{X}_A)|_q$, $\mathscr{L}_{NS}(A\tilde{Y}_A)|_q$ and $\mathscr{L}_{NS}(\hat{Z}_A)|_q$ in the last vectors columns that if one of the following conditions is satisfied

$$\begin{aligned} &1)\ \ G_{\tilde{X}} \neq 0, \\ &2)\ \ \omega G_{\tilde{Y}} - H_{\tilde{X}} \neq 0, \\ &3)\ \ H_{\tilde{Y}} \neq 0, \\ &4)\ \ \omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi - 1 \neq 0, \\ &5)\ \ \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\omega \neq 0. \end{aligned} \tag{5.31}$$

then $\mathcal{O}_{\mathscr{D}_R}(q_0)$ have dimension great or equal to 6. However, the negative case of (5.31), i.e.

$$G_{\tilde{X}} \equiv 0,\ \omega G_{\tilde{Y}} \equiv H_{\tilde{X}},\ H_{\tilde{Y}} \equiv 0,\ \omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi \equiv 1,\ \text{and}\ \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\omega \equiv 0, \tag{5.32}$$

give us the next result.

Proposition 5.3.19. *Under the assumption* (5.32)*, we have* $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 5$*,* M *is a flat manifold and* $\hat{M}$ *is (locally) a Riemannien product of a real interval with 2-dimensional totally geodesic submanifold.*

Proof. (5.32)), (5.24), (5.25) and (5.28) give us the following results

$$\begin{aligned}
&\text{since } \omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi \equiv 1, \text{ we get } \tilde{\hat{\sigma}}_A^2 \equiv K,\\
&\text{since } \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\omega \equiv 0, \text{ we get } \tilde{\Pi}_{\hat{Z}} = 0,\\
&0 = \nu(\overline{\mathsf{Rol}}_q)|_q\tilde{\Pi}_{\hat{Z}} = (\tilde{\hat{\sigma}}_A^1 - K) + \omega^2(K - \hat{\sigma}_A),\\
&\nu(\overline{\mathsf{Rol}}_q)|_q\phi = 0, \nu(\overline{\mathsf{Rol}}_q)|_q\omega = 0, F_1|_q\phi = 0,\\
&\text{computing } [\mathscr{L}_R(\tilde{Y}), \nu(\overline{\mathsf{Rol}}_{(\cdot)})]|_q\omega, \text{ we get } F_2|_q\omega = 0,\\
&\text{computing } [\mathscr{L}_R(\tilde{X}), \mathscr{L}_R(\tilde{Y})]|_q\phi, \text{ we get } \mathscr{L}_R(\tilde{X}_A)|_q G_{\tilde{Y}} = -(G_{\tilde{Y}})^2 - K,\\
&\text{computing } [\mathscr{L}_R(\tilde{X}), \mathscr{L}_R(\tilde{Y})]|_q\omega, \text{ we get } \mathscr{L}_R(\tilde{Y}_A)|_q H_{\tilde{X}} = \omega\mathscr{L}_R(\tilde{Y}_A)|_q G_{\tilde{Y}} = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\text{computing } [\mathscr{L}_R(\tilde{X}), \nu(\overline{\mathsf{Rol}}_{(\cdot)})]|_q\omega, \text{ we get } \omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q(G_{\tilde{Y}}) = -F_1|_q\omega,\\
&\text{computing } [\mathscr{L}_R(\tilde{Y}), \nu(\overline{\mathsf{Rol}}_{(\cdot)})]|_q\phi, \text{ we get } \nu(\overline{\mathsf{Rol}}_q)|_q(G_{\tilde{Y}}) = -F_2|_q\phi,
\end{aligned}$$

However $[\nu(\overline{\mathsf{Rol}}_{(\cdot)}), F_1]|_q\phi = 0$, then $F_2|_q\phi = 0$. This implies

$$\omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q(G_{\tilde{Y}}) = -F_1|_q\omega = -\omega F_2|_q\phi = 0.$$

On the other hand,

$$\begin{aligned}
&\text{computing } [\mathscr{L}_R(\tilde{X}), F_1]|_q\omega, \text{ we get } F_1|_q H_{\tilde{X}} = 0,\\
&\text{computing } [\mathscr{L}_R(\tilde{X}), F_2]|_q\omega, \text{ we get } F_2|_q H_{\tilde{X}} = 0.
\end{aligned}$$

All these results prove that the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ at point $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ is generated by the vectors:

$$\mathscr{L}_R(\tilde{X}_A)|_q, \mathscr{L}_R(\tilde{Y}_A)|_q, F_1|_q, F_2|_q, \nu(\overline{\mathsf{Rol}}_q)|_q.$$

□

Proposition 5.3.20. *Under the assumption* (5.32)*,* M *is a flat manifold.*

Proof. Referring to the previous proposition, the matrix of $\hat{R}$ with respect to the base $\star A\tilde{X}_A$, $\star A\tilde{Y}_A$, $\star\hat{Z}_A$,

$$\begin{pmatrix} -\tilde{\hat{\sigma}}_A^1 & 0 & \omega(K - \hat{\sigma}_A) \\ 0 & -K & 0 \\ \omega(K - \hat{\sigma}_A) & 0 & -\hat{\sigma}_A \end{pmatrix}.$$

By the equality $\omega^2(K-\hat{\sigma}_A) = K - \tilde{\hat{\sigma}}^1_A$, the characteristic polynomial f of $\hat{R}$ is

$$f(\tau) = -(\tau+K)^2(\tau + \tilde{\hat{\sigma}}^1_A + \hat{\sigma}_A - K).$$

Then, $-K$ is a double eigenvalue of $\hat{R}$ where its two eigenvectors are $M_1 := -\star A\tilde{X}_A + \omega\star\hat{Z}_A$ and $M_2 := \star A\tilde{Y}_A$. The third eigenvalue of $\hat{R}$ is $\lambda := -\tilde{\hat{\sigma}}^1_A - \hat{\sigma}_A + K$ and its corresponding eigenvector is $M_3 := -\omega\star A\tilde{X}_A - \star\hat{Z}_A$.

Further more, F_1 and F_2 are now simplified to

$$\begin{aligned} F_1|_q =& -\mathscr{L}_{NS}(A\tilde{Y}_A)|_q + H_{\tilde{X}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q, \\ F_2|_q =& \mathscr{L}_{NS}(A\tilde{X}_A)|_q - \omega\mathscr{L}_{NS}(\hat{Z}_A)|_q - H_{\tilde{X}}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q. \end{aligned}$$

Therefore, we have the following tangent Lie brackets to $\mathcal{O}_{\mathscr{D}_R}(q_0)$

$$\begin{aligned} [\mathscr{L}_R(\tilde{X}), F_1]|_q =& -\hat{\sigma}_A\nu(\overline{\mathsf{Rol}}_q)|_q, \\ [\mathscr{L}_R(\tilde{Y}), F_1]|_q =& G_{\tilde{Y}}F_2|_q, \\ [\nu(\overline{\mathsf{Rol}}_{(\cdot)}), F_1]|_q =& F_2|_q, \\ [\mathscr{L}_R(\tilde{X}), F_2]|_q =& 0, \\ [\mathscr{L}_R(\tilde{Y}), F_2]|_q =& -G_{\tilde{Y}}F_1|_q + \lambda\nu(\overline{\mathsf{Rol}}_q)|_q, \\ [\nu(\overline{\mathsf{Rol}}_{(\cdot)}), F_2]|_q =& -(1+\omega^2)F_1|_q + \omega H_{\tilde{X}}\nu(\overline{\mathsf{Rol}}_q)|_q, \\ [F_1, F_2]|_q =& -\omega H_{\tilde{X}}F_1|_q + ((H_{\tilde{X}})^2 - \lambda)\nu(\overline{\mathsf{Rol}}_q)|_q. \end{aligned}$$

Apply the Jacobi identity with the three vector fields $\mathscr{L}_R(\tilde{Y}_A)|_q$, $F_1|_q$, $F_2|_q$, i.e.

$$[\mathscr{L}_R(\tilde{Y}), [F_1, F_2]]|_q + [F_2, [\mathscr{L}_R(\tilde{Y}), F_1]]|_q + [F_1, [F_2, \mathscr{L}_R(\tilde{Y})]]|_q = 0. \tag{5.33}$$

The first bracket of the previous equality is equal to

$$\begin{aligned} [\mathscr{L}_R(\tilde{Y}), [F_1, F_2]]|_q =& [\mathscr{L}_R(\tilde{Y}), -\omega H_{\tilde{X}}F_1 + ((H_{\tilde{X}})^2 - \lambda)\nu(\overline{\mathsf{Rol}}_{(\cdot)})]|_q \\ =& -(\mathscr{L}_R(\tilde{Y}_A)|_q\lambda)\nu(\overline{\mathsf{Rol}}_q)|_q - \lambda F_2|_q, \end{aligned}$$

the second one is equal to

$$[F_2, [\mathscr{L}_R(\tilde{Y}), F_1]]|_q = [F_2, G_{\tilde{Y}}F_2]|_q = 0,$$

and the last one is equal to

$$[F_1, [F_2, \mathscr{L}_R(\tilde{Y})]]|_q = [F_1, G_{\tilde{Y}}F_1 - \lambda\nu(\overline{\mathsf{Rol}}_{(\cdot)})]|_q = \lambda F_2|_q.$$

Replace the brackets in (5.33) by they values, we obtain that

$$\mathscr{L}_R(\tilde{Y}_A)|_q\lambda = 0.$$

This implies that

$$\tilde{Y}_A(K) = \hat{D}_{A\tilde{Y}_A}(\tilde{\hat{\sigma}}^1_A) + \hat{D}_{A\tilde{Y}_A}(\hat{\sigma}_A).$$

However, we have

$$0 = F_1|_q\omega = -\frac{\omega\tilde{Y}_A(K)}{(K-\hat{\sigma}_A)} + H_{\tilde{X}}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\omega = -\frac{\omega\tilde{Y}_A(K)}{(K-\hat{\sigma}_A)}.$$

Then

$$\tilde{Y}_A(K) = 0, \tag{5.34}$$

and,

$$\hat{D}_{A\tilde{Y}_A}(\tilde{\hat{\sigma}}^1_A) + \hat{D}_{A\tilde{Y}_A}(\hat{\sigma}_A) = 0. \tag{5.35}$$

Apply again the Jacobi identity with the vectors fields $\mathscr{L}_R(\tilde{X}_A)|_q$, $F_1|_q$, $F_2|_q$ and use the same method as before, we obtain

$$\mathscr{L}_R(\tilde{X}_A)|_q\lambda = 2\omega H_{\tilde{X}}(\hat{\sigma}_A - K) - F_2|_q\hat{\sigma}_A,$$

i.e.

$$\tilde{X}_A(K) = \hat{D}_{A\tilde{X}_A}(\tilde{\hat{\sigma}}^1_A) + \omega\hat{D}_{\hat{Z}_A}(\hat{\sigma}_A). \tag{5.36}$$

On the other hand, the second Bianchi identity allows us to have

$$\hat{D}_{A\tilde{X}_A}(\tilde{\hat{\sigma}}^1_A) = \hat{D}_{A\tilde{Y}_A}(\tilde{\Pi}_{\hat{Z}}) + \hat{D}_{\hat{Z}_A}(\Pi_{\tilde{X}_A}). \tag{5.37}$$

(5.34), (5.35), (5.36), (5.37) and the derivation of $\Pi_{\tilde{X}_A} = \omega(K-\hat{\sigma}_A)$ along $\mathscr{L}_R(\tilde{X}_A)$, $\mathscr{L}_R(\tilde{Y}_A)$, F_2 give

$$\omega\tilde{X}_A(K) = \hat{D}_{A\tilde{X}_A}(\Pi_{\tilde{X}_A}) + \omega\hat{D}_{A\tilde{X}_A}(\hat{\sigma}_A) - H_{\tilde{X}}(K-\hat{\sigma}_A), \tag{5.38}$$
$$\hat{D}_{A\tilde{X}_A}(\tilde{\hat{\sigma}}^1_A) = \hat{D}_{\hat{Z}_A}(\Pi_{\tilde{X}_A}) + \omega H_{\tilde{X}}(K-\hat{\sigma}_A), \tag{5.39}$$
$$\hat{D}_{A\tilde{Y}_A}(\Pi_{\tilde{X}_A}) = -\omega\hat{D}_{A\tilde{Y}_A}(\hat{\sigma}_A) = \omega\hat{D}_{A\tilde{Y}_A}(\tilde{\hat{\sigma}}^1_A), \tag{5.40}$$
$$\hat{D}_{A\tilde{Y}_A}(\tilde{\Pi}_{\hat{Z}}) = \omega H_{\tilde{X}}(K-\hat{\sigma}_A). \tag{5.41}$$

Differentiating $\Pi_{\tilde{Y}_A} = 0$ with respect to $\mathscr{L}_R(\tilde{X}_A)$, $\mathscr{L}_R(\tilde{Y}_A)$, F_2 yield

$$\hat{D}_{A\tilde{X}_A}(\Pi_{\tilde{Y}_A}) = 0, \tag{5.42}$$
$$\hat{D}_{A\tilde{Y}_A}(\Pi_{\tilde{Y}_A}) = H_{\tilde{X}}(K-\hat{\sigma}_A), \tag{5.43}$$
$$\hat{D}_{\hat{Z}_A}(\Pi_{\tilde{Y}_A}) = 0. \tag{5.44}$$

Differentiating $\tilde{\Pi}_{\hat{Z}} = 0$ with respect to $\mathscr{L}_R(\tilde{X}_A)$ yield

$$\hat{D}_{A\tilde{X}_A}(\tilde{\Pi}_{\hat{Z}}) = 0. \tag{5.45}$$

(5.36) and the derivative of $\tilde{\tilde{\sigma}}^2_A = K$ with respect to $\mathscr{L}_R(\tilde{X}_A)$,$F_2$ give

$$\tilde{X}_A(K) = \hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^2_A), \tag{5.46}$$
$$\hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^2_A) = \hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^1_A) + \omega \hat{D}_{\hat{Z}_A}(\hat{\sigma}_A), \tag{5.47}$$
$$\hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^2_A) = \omega \hat{D}_{\hat{Z}_A}(\tilde{\tilde{\sigma}}^2_A), \tag{5.48}$$
$$\omega \hat{D}_{\hat{Z}_A}(\tilde{\tilde{\sigma}}^2_A) = \hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^1_A) + \omega \hat{D}_{\hat{Z}_A}(\hat{\sigma}_A). \tag{5.49}$$

(5.38), (5.43), (5.46) and the derivative of (5.42) with respect to $\nu(\overline{\mathsf{Rol}}_q)$ give

$$\hat{D}_{A\tilde{X}_A}(\hat{\sigma}_A) = 0. \tag{5.50}$$

(5.39), (5.43), (5.49) and the derivative of (5.44) with respect to $\nu(\overline{\mathsf{Rol}}_q)$ give

$$\hat{D}_{\hat{Z}_A}(\hat{\sigma}_A) = 0. \tag{5.51}$$

Thus, by (5.47), we obtain

$$\hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^1_A) = \hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^2_A). \tag{5.52}$$

(5.38), (5.41), (5.50), (5.52) and the derivative of (5.45) with respect to $\nu(\overline{\mathsf{Rol}}_q)$ give

$$\tilde{X}_A(K) = 0. \tag{5.53}$$

So, (5.34) and (5.53) implies that M has a constant curvature K. We also get

$$\hat{D}_{A\tilde{X}_A}(\tilde{\tilde{\sigma}}^1_A) = 0, \tag{5.54}$$

so, by (5.39),

$$\hat{D}_{\hat{Z}_A}(\Pi_{\tilde{X}_A}) = -\omega H_{\tilde{X}}(K - \hat{\sigma}_A). \tag{5.55}$$

(5.45) and the derivative of (5.54) with respect to $\nu(\overline{\mathsf{Rol}}_q)$ give

$$\hat{D}_{A\tilde{Y}_A}(\tilde{\tilde{\sigma}}^1_A) = 0. \tag{5.56}$$

Then we obtain, by (5.40)

$$\hat{D}_{A\tilde{Y}_A}(\Pi_{\tilde{X}_A}) = 0. \tag{5.57}$$

Differentiating (5.57) with respect to $\nu(\overline{\mathsf{Rol}}_q)$ and using (5.38), (5.41), (5.43), (5.55) yield

$$H_{\tilde{X}} \equiv 0.$$

Therefore,

$$G_{\tilde{Y}} \equiv 0,$$

and,

$$0 = \mathscr{L}_R(\tilde{X}_A)|_q G_{\tilde{Y}} = -K,$$

then M is a flat manifold. □

Proposition 5.3.21. *Under the assumption* (5.32), $\hat{M}$ *is (locally) a Riemannien product of a real interval with 2-dimensional totally geodesic submanifold.*

Proof. From the proof of Proposition 5.3.21, 0 is a double eigenvalue of $\hat{R}$ with its corresponding eigenvectors M_1 and M_2. Since we now have $\tilde{\hat{\sigma}}_A^1 = \omega^2\hat{\sigma}_A$, then the third eigenvalue λ of $\hat{R}$ is equal to $-\hat{\sigma}_A(1+\omega^2)$ with the corresponding eigenvector M_3. The matrix form of $\hat{R}$ is

$$-\hat{\sigma}_A\begin{pmatrix} \omega^2 & 0 & \omega \\ 0 & 0 & 0 \\ \omega & 0 & 1 \end{pmatrix}.$$

We may assume without loss of generality that $\hat{R}$ cannot be a null symmetric matrix, i.e. $\hat{\sigma}_A$ is not vanish on $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Define a local orthonormal frame $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$ on $\hat{M}$. Let K_1, K_2, K_3 be the eigenvalues of $\hat{R}$ corresponding to $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$ respectively and assume that $K_3 \neq 0$. Since $\lambda \neq 0$, then $\star\hat{E}_3 = \pm\frac{1}{\sqrt{1+\omega^2}}M_3$ and hence $span\{\hat{E}_1, \hat{E}_2\} = span\{\star M_1, \star M_2\}$. On the other hand, we have that $\nu(\mathsf{Rol}_q)|_q = -\nu(M_3A)|_q = \mp\sqrt{(1+\omega^2)}\nu(\star\hat{E}_3A)|_q$ then $\nu(\star\hat{E}_3A)|_q$ is tangent to the orbit. Further more, F_1 and F_2 are now equal to

$$\begin{aligned} F_1|_q =& -\mathscr{L}_{NS}(A\tilde{Y}_A)|_q, \\ F_2|_q =& \mathscr{L}_{NS}(A\tilde{X}_A)|_q - \omega\mathscr{L}_{NS}(\hat{Z}_A)|_q. \end{aligned}$$

Thus, $F_1|_q = -\mathscr{L}_{NS}(\star M_2)|_q$ and $F_2|_q = -\mathscr{L}_{NS}(\star M_1)|_q$. Therefore, $\mathscr{L}_{NS}(\hat{E}_1)|_q$ and $\mathscr{L}_{NS}(\hat{E}_2)|_q$ are tangents to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Then one can treat the matter of the matrix form of the connection $\hat{\Gamma}$ as in the proof of Corollary 5.3.14. Thus,

$$\hat{\Gamma} = \begin{pmatrix} 0 & 0 & \hat{\Gamma}^3_{(2,3)} \\ 0 & 0 & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & \hat{\Gamma}^2_{(1,2)} & \hat{\Gamma}^3_{(1,2)} \end{pmatrix}.$$

Thanks to (5.63), we have the following formulas

$$\begin{array}{lr} \hat{E}_1(\hat{\Gamma}^2_{(1,2)}) - \hat{E}_2(\hat{\Gamma}^1_{(1,2)}) + (\hat{\Gamma}^1_{(1,2)})^2 + (\hat{\Gamma}^2_{(1,2)})^2 = -\hat{\sigma}_A(1+\omega^2), & (5.58)-(1) \\ \hat{E}_2(\hat{\Gamma}^3_{(2,3)}) - \hat{\Gamma}^2_{(1,2)}\hat{\Gamma}^3_{(3,1)} + (\hat{\Gamma}^3_{(2,3)})^2 = 0, & (5.58)-(2) \\ \hat{E}_2(\hat{\Gamma}^3_{(3,1)}) + \hat{\Gamma}^3_{(3,1)}\hat{\Gamma}^3_{(2,3)} + \hat{\Gamma}^2_{(1,2)}\hat{\Gamma}^3_{(2,3)} = 0, & (5.58)-(3) \\ \hat{E}_1(\hat{\Gamma}^3_{(2,3)}) - \hat{\Gamma}^3_{(2,3)}\hat{\Gamma}^3_{(3,1)} - \hat{\Gamma}^1_{(1,2)}\hat{\Gamma}^3_{(3,1)} = 0, & (5.58)-(4) \\ \hat{E}_1(\hat{\Gamma}^3_{(3,1)}) + \hat{\Gamma}^3_{(2,3)}\hat{\Gamma}^1_{(1,2)} - (\hat{\Gamma}^3_{(3,1)})^2 = 0. & (5.58)-(5) \end{array} \tag{5.58}$$

We are looking to prove that $\hat{\Gamma}^3_{(3,1)} = 0$ and $\hat{\Gamma}^3_{(2,3)} = 0$. Indeed, we have

$$[\hat{E}_1, \hat{E}_2]|_{\hat{x}} = -\hat{\Gamma}^1_{(1,2)}(\hat{x})\hat{E}_1|_{\hat{x}} - \hat{\Gamma}^2_{(1,2)}(\hat{x})\hat{E}_2|_{\hat{x}}. \tag{5.59}$$

The integrability relation of this Lie bracket on $\hat{\Gamma}^3_{(3,1)}$ and (5.58)-(5) allows us to have

$$\hat{\Gamma}^3_{(2,3)} = 0.$$

We proceed by absurd to prove that $\hat{\Gamma}^3_{(3,1)} = 0$, i.e. suppose that $\hat{\Gamma}^3_{(3,1)}(\hat{x}) \neq 0$. Since $\hat{\Gamma}^3_{(2,3)} = 0$, (5.58)-(2) and (5.58)-(4) imply that $\hat{\Gamma}^1_{(1,2)} = 0$ and $\hat{\Gamma}^2_{(1,2)} = 0$, then (5.58)-(1) will imply that $\hat{\sigma}_A = 0$, which it is a contradiction. Then,

$$\hat{\Gamma}^3_{(3,1)} = 0.$$

and hence the matrix form of $\hat{\Gamma}$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hat{\Gamma}^1_{(1,2)} & \hat{\Gamma}^2_{(1,2)} & \hat{\Gamma}^3_{(1,2)} \end{pmatrix}.$$

Denoted $\hat{N}$ the submanifold of $\hat{M}$ spanned by $\hat{E}_1$ and $\hat{E}_2$. $\hat{N}$ has $\hat{E}_3$ as a normal vector. If we adapt the above results with the assumptions of Theorem 5.4.4, we conclude that there is an interval $I \subset \mathbb{R}$ such that $(\hat{V}, g|_{\hat{V}})$ is isometric to the Riemannian product $(I \times \hat{N}, \hat{g})$. Moreover, one can easily check that $(\hat{\nabla}_{\hat{E}_i} \hat{R})\hat{E}_3 = 0$ for $i = 1, 2$. Therefore $\hat{N}$ is a totally geodesic submanifold of $\hat{M}$. □

Returning to the conditions in (5.31), assume that there is one of them is realize. We will obtain from the column vectors that the following vector fields forms are tangents to $\mathcal{O}_{\mathscr{D}_R}(q_0)$

$$\begin{aligned} L_1 &:= \mathscr{L}_{NS}(A\tilde{X}_A)|_q + \alpha_1 \nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q + \beta_1 \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q, \\ L_2 &:= \mathscr{L}_{NS}(A\tilde{Y}_A)|_q + \alpha_2 \nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q + \beta_2 \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q, \\ L_3 &:= \mathscr{L}_{NS}(\hat{Z}_A)|_q + \alpha_3 \nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q + \beta_3 \nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q, \end{aligned}$$

where α_i, β_i, $i = 1, 2, 3$, are functions on Q. Since, we are looking to find the necessary conditions for the controllability, then we assume that the dimension of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is less or equal to 7. Then in this situation, we must have $\dim \mathcal{O}_{\mathscr{D}_R}(q_0)$ is equal to 6 or 7. Actually, we didn't find the final solution, but we will present our initial calculation.

Dim $\mathcal{O}_{\mathscr{D}_R}(q_0) = 6$: Here, we suppose that $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 6$, i.e. $T_q\mathcal{O}_{\mathscr{D}_R}(q_0)$, for a $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, is generated by

$$\mathscr{L}_R(\tilde{X}_A)|_q, \mathscr{L}_R(\tilde{Y}_A)|_q, \nu(\mathsf{Rol}_q)|_q, L_1|_q, L_2|_q, L_3|_q.$$

On the other hand, F_1 and F_2 are tangents to $\mathcal{O}_{\mathscr{D}_R}(q_0)$, thus one can remark that $L_1 - \omega L_3 = F_2$ and $L_2 = -F_1$ i.e.

$$\alpha_1 - \omega\alpha_3 = -\omega G_{\tilde{Y}}, \quad \beta_1 - \omega\beta_3 = H_{\tilde{Y}}, \quad \alpha_2 = \omega G_{\tilde{X}}, \quad \beta_2 = -H_{\tilde{X}}. \tag{5.60}$$

Lemma 5.3.22. *Differentiating α_i, β_i for $i = 1, 2, 3$ with respect to $\mathscr{L}_R(\tilde{X}_A)|_q$, $\mathscr{L}_R(\tilde{Y}_A)|_q$*

and $\nu(\overline{\mathsf{Rol}}_q)|_q$ *yields*

$$\begin{aligned}
\mathscr{L}_R(\tilde{X}_A)|_q\alpha_1 &= \beta_1 G_{\tilde{X}} + \alpha_2 G_{\tilde{X}} - \alpha_1\alpha_3,\\
\mathscr{L}_R(\tilde{X}_A)|_q\beta_1 &= -\alpha_1 G_{\tilde{X}} + \beta_2 G_{\tilde{X}} - \alpha_1\beta_3,\\
\mathscr{L}_R(\tilde{X}_A)|_q\alpha_2 &= \beta_2 G_{\tilde{X}} - \alpha_1 G_{\tilde{X}} - \alpha_2\alpha_3,\\
\mathscr{L}_R(\tilde{X}_A)|_q\beta_2 &= -\alpha_2 G_{\tilde{X}} - \beta_1 G_{\tilde{X}} - \alpha_2\beta_3 + \omega K,\\
\mathscr{L}_R(\tilde{X}_A)|_q\alpha_3 &= \beta_3 G_{\tilde{X}} - \alpha_3^2 - \tilde{\hat{\sigma}}_A^2,\\
\mathscr{L}_R(\tilde{X}_A)|_q\beta_3 &= -\alpha_3 G_{\tilde{X}} - \alpha_3\beta_3 - \tilde{\Pi}_{\hat{Z}},
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_R(\tilde{Y}_A)|_q\alpha_1 &= \beta_1 G_{\tilde{Y}} + \alpha_2 G_{\tilde{Y}} - \alpha_3\beta_1,\\
\mathscr{L}_R(\tilde{Y}_A)|_q\beta_1 &= -\alpha_1 G_{\tilde{Y}} + \beta_2 G_{\tilde{Y}} - \beta_1\beta_3 - \omega K,\\
\mathscr{L}_R(\tilde{Y}_A)|_q\alpha_2 &= \beta_2 G_{\tilde{Y}} - \alpha_1 G_{\tilde{Y}} - \alpha_3\beta_2,\\
\mathscr{L}_R(\tilde{Y}_A)|_q\beta_2 &= -\alpha_2 G_{\tilde{Y}} - \beta_1 G_{\tilde{Y}} - \beta_2\beta_3,\\
\mathscr{L}_R(\tilde{Y}_A)|_q\alpha_3 &= \beta_3 G_{\tilde{Y}} - \alpha_3\beta_3 - \tilde{\Pi}_{\hat{Z}},\\
\mathscr{L}_R(\tilde{Y}_A)|_q\beta_3 &= -\alpha_3 G_{\tilde{Y}} - \beta_3^2 - \omega^2(K - \hat{\sigma}_A) - \tilde{\hat{\sigma}}_A^1.
\end{aligned}$$

$$\begin{aligned}
\nu(\overline{\mathsf{Rol}}_q)|_q\alpha_1 &= -\omega\alpha_1\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\phi + \omega\alpha_2\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi - \omega G_{\tilde{X}},\\
\nu(\overline{\mathsf{Rol}}_q)|_q\beta_1 &= -\omega\alpha_1\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi + \omega\beta_2\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi + L_1|_q\omega - \omega^2\alpha_1,\\
\nu(\overline{\mathsf{Rol}}_q)|_q\alpha_2 &= -\omega\alpha_2\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\phi - \omega\alpha_1\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi - \omega G_{\tilde{Y}} + \omega\alpha_3,\\
\nu(\overline{\mathsf{Rol}}_q)|_q\beta_2 &= -\omega\alpha_2\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi - \omega\beta_1\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi + L_2|_q\omega - \omega^2\alpha_2 + \omega\beta_3,\\
\nu(\overline{\mathsf{Rol}}_q)|_q\alpha_3 &= -\omega\alpha_3\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\phi - \omega\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi - \omega\alpha_2,\\
\nu(\overline{\mathsf{Rol}}_q)|_q\beta_3 &= -\omega\alpha_3\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi + L_3|_q\omega - \omega^2\alpha_3 - \omega\beta_2.
\end{aligned}$$

Proof. Computing the Lie brackets on $\mathcal{O}_{\mathscr{D}_R}(q_0)$, we get,

$$\begin{aligned}
& [\mathscr{L}_R(\tilde{X}), L_1]|_q\\
= \ & -(G_{\tilde{X}} + \alpha_1\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\phi + \beta_1\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q\\
& +G_{\tilde{X}}L_2|_q - \alpha_1 L_3|_q\\
& +(\mathscr{L}_R(\tilde{X}_A)|_q\alpha_1 - \beta_1 G_{\tilde{X}} - \alpha_2 G_{\tilde{X}} + \alpha_1\alpha_3)\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\\
& +(\mathscr{L}_R(\tilde{X}_A)|_q\beta_1 + \alpha_1 G_{\tilde{X}} - \beta_2 G_{\tilde{X}} + \alpha_1\beta_3)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$\begin{aligned}
& [\mathscr{L}_R(\tilde{X}), L_2]|_q\\
= \ & -(G_{\tilde{Y}} + \alpha_2\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\phi + \beta_2\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q\\
& +\hat{\sigma}_A\nu(\overline{\mathsf{Rol}}_q)|_q - G_{\tilde{X}}L_1|_q - \alpha_2 L_3|_q\\
& +(\mathscr{L}_R(\tilde{X}_A)|_q\alpha_2 - \beta_2 G_{\tilde{X}} + \alpha_1 G_{\tilde{X}} + \alpha_2\alpha_3)\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q\\
& +(\mathscr{L}_R(\tilde{X}_A)|_q\beta_2 + \alpha_2 G_{\tilde{X}} + \beta_1 G_{\tilde{X}} + \alpha_2\beta_3 - \omega K)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{X}), L_3]|_q \\
= \; & -(\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi + \alpha_3\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi + \beta_3\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{Y}_A)|_q \\
& -\alpha_3 L_3|_q \\
& +(\mathscr{L}_R(\tilde{X}_A)|_q\alpha_3 - \beta_3 G_{\tilde{X}} + \alpha_3^2 + \tilde{\hat{\sigma}}_A^2)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
& +(\mathscr{L}_R(\tilde{X}_A)|_q\beta_3 + \alpha_3 G_{\tilde{X}} + \alpha_3\beta_3 + \tilde{\Pi}_{\hat{Z}})\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), L_1]|_q \\
= \; & (G_{\tilde{X}} + \alpha_1\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi + \beta_1\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q \\
& +G_{\tilde{Y}} L_2|_q - \beta_1 L_3|_q - \hat{\sigma}_A\nu(\overline{\mathsf{Rol}_q})|_q \\
& +(\mathscr{L}_R(\tilde{Y}_A)|_q\alpha_1 - \beta_1 G_{\tilde{Y}} - \alpha_2 G_{\tilde{Y}} + \alpha_3\beta_1)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
& +(\mathscr{L}_R(\tilde{Y}_A)|_q\beta_1 + \alpha_1 G_{\tilde{Y}} - \beta_2 G_{\tilde{Y}} + \beta_1\beta_3 + \omega K)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), L_2]|_q \\
= \; & (G_{\tilde{Y}} + \alpha_2\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi + \beta_2\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q \\
& -G_{\tilde{Y}} L_1|_q - \beta_2 L_3|_q \\
& +(\mathscr{L}_R(\tilde{Y}_A)|_q\alpha_2 - \beta_2 G_{\tilde{Y}} + \alpha_1 G_{\tilde{Y}} + \alpha_3\beta_2)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
& +(\mathscr{L}_R(\tilde{Y}_A)|_q\beta_2 + \alpha_2 G_{\tilde{Y}} + \beta_1 G_{\tilde{Y}} + \beta_2\beta_3)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(\tilde{Y}), L_3]|_q \\
= \; & (\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi + \alpha_3\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi + \beta_3\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\mathscr{L}_R(\tilde{X}_A)|_q \\
& -\beta_3 L_3|_q - \omega(K - \hat{\sigma}_A)\nu(\overline{\mathsf{Rol}_q})|_q \\
& +(\mathscr{L}_R(\tilde{Y}_A)|_q\alpha_3 - \beta_3 G_{\tilde{Y}} + \alpha_3\beta_3 + \tilde{\Pi}_{\hat{Z}})\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
& +(\mathscr{L}_R(\tilde{Y}_A)|_q\beta_3 + \alpha_3 G_{\tilde{Y}} + \beta_3^2 + \omega^2(K - \hat{\sigma}_A) + \tilde{\hat{\sigma}}_A^1)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\nu(\overline{\mathsf{Rol}_{(\cdot)}}), L_1] \\
= \; & (\omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)L_2|_q - \omega\alpha_1\nu(\overline{\mathsf{Rol}_q})|_q \\
& +(\nu(\overline{\mathsf{Rol}_q})|_q\alpha_1 + \omega G_{\tilde{X}} + \omega\alpha_1\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi - \omega\alpha_2\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
& +(\nu(\overline{\mathsf{Rol}_q})|_q\beta_1 + \omega^2\alpha_1 + \omega(\alpha_1 - \beta_2)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi - L_1|_q\omega)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\nu(\overline{\mathsf{Rol}_{(\cdot)}}), L_2] \\
= \; & -(\omega\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)L_1|_q + \omega L_3|_q - \omega\alpha_2\nu(\overline{\mathsf{Rol}_q})|_q \\
& +(\nu(\overline{\mathsf{Rol}_q})|_q\alpha_2 + \omega G_{\tilde{Y}} - \omega\alpha_3 + \omega\alpha_2\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi + \omega\alpha_1\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
& +(\nu(\overline{\mathsf{Rol}_q})|_q\beta_2 + \omega^2\alpha_2 - \omega\beta_3 + \omega(\alpha_2 + \beta_1)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi - L_2|_q\omega)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\nu(\overline{\mathsf{Rol}_{(\cdot)}}), L_3] \\
= \; & -\omega L_2|_q - \omega\alpha_3\nu(\overline{\mathsf{Rol}_q})|_q \\
& +(\nu(\overline{\mathsf{Rol}_q})|_q\alpha_3 + \omega\alpha_2 + \omega\alpha_3\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q\phi + \omega\mathscr{L}_{NS}(\hat{Z}_A)|_q\phi)\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q \\
& +(\nu(\overline{\mathsf{Rol}_q})|_q\beta_3 + \omega^2\alpha_3 + \omega\beta_2 + \omega\alpha_3\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q\phi - L_3|_q\omega)\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q.
\end{aligned}
$$

Since $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 6$, all the coefficients of $\nu(\theta_{\tilde{X}_A} \otimes \hat{Z}_A)|_q$ and $\nu(\theta_{\tilde{Y}_A} \otimes \hat{Z}_A)|_q$ in the previous brackets are equal to zero, then we get the claim. □

Remark 5.3.23. Using integrability relation of the bracket $[\mathscr{L}_R(\tilde{X}), \mathscr{L}_R(\tilde{Y})]$ on ϕ and

ω, we find

$$\begin{aligned}
&\mathscr{L}_R(\tilde{X}_A)|_q G_{\tilde{Y}} - \mathscr{L}_R(\tilde{Y}_A)|_q G_{\tilde{X}} = -G_{\tilde{X}}^2 - G_{\tilde{Y}}^2 - \tilde{\hat{\sigma}}_A^2, \\
&\mathscr{L}_R(\tilde{X}_A)|_q H_{\tilde{Y}} - \mathscr{L}_R(\tilde{Y}_A)|_q H_{\tilde{X}} = -G_{\tilde{X}} H_{\tilde{X}} - G_{\tilde{Y}} H_{\tilde{Y}} + \omega \tilde{\Pi}_{\hat{Z}}.
\end{aligned}$$

Lemma 5.3.24. *We have the following formulas on the derivatives of* α_i, β_i *for* $i = 1,2,3$ *with respect to* $L_1|_q$, $L_2|_q$ *and* $L_3|_q$,

$$L_1|_q\alpha_2 - L_2|_q\alpha_1 = (-\alpha_1+\beta_2)L_1|_q\phi - (\beta_1+\alpha_2)L_2|_q\phi + \beta_1\alpha_3 - \alpha_2\alpha_3,$$

$$\begin{aligned}
& L_1|_q\beta_2 - L_2|_q\beta_1 \\
= \ & -(\beta_1+\alpha_2)L_1|_q\phi + (\alpha_1-\beta_2)L_2|_q\phi + \beta_3\beta_1 - \alpha_2\beta_3 + (\alpha_1\beta_2 - \beta_1\alpha_2)\omega + \omega K.
\end{aligned}$$

$$L_2|_q\alpha_3 - L_3|_q\alpha_2 = \beta_3 L_2|_q\phi + (\alpha_1-\beta_2)L_3|_q\phi - \alpha_1\alpha_2 - \alpha_2\beta_2 - \alpha_3\beta_3 - \tilde{\Pi}_{\hat{Z}},$$

$$\begin{aligned}
& L_2|_q\beta_3 - L_3|_q\beta_2 \\
= \ & -\alpha_3 L_2|_q\phi + (\alpha_2+\beta_1)L_3|_q\phi - \beta_1\alpha_2 - \beta_2^2 - \beta_3^2 + (\alpha_2\beta_3 - \beta_2\alpha_3)\omega - \omega\Pi_{\tilde{X}} - \tilde{\hat{\sigma}}_A^1.
\end{aligned}$$

$$L_3|_q\alpha_1 - L_1|_q\alpha_3 = (\alpha_2+\beta_1)L_3|_q\phi - \beta_3 L_1|_q\phi + \alpha_1^2 + \alpha_2\beta_1 + \alpha_3^2 + \tilde{\hat{\sigma}}_A^2,$$

$$\begin{aligned}
& L_3|_q\beta_1 - L_1|_q\beta_3 \\
= \ & (-\alpha_1+\beta_2)L_3|_q\phi + \alpha_3 L_1|_q\phi + \alpha_1\beta_1 + \beta_1\beta_2 + \alpha_3\beta_3 - (\alpha_1\beta_3 - \beta_1\alpha_3)\omega + \tilde{\Pi}_{\hat{Z}}.
\end{aligned}$$

Proof. Computing on the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ at $q = (x,\hat{x};A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$,

$$\begin{aligned}
& [L_1, L_2]|_q \\
= & -(L_1|_q\phi)L_1|_q - (L_2|_q\phi)L_2|_q + (\beta_1-\alpha_2)L_3|_q + (\alpha_1\beta_2 - \beta_1\alpha_2 + \hat{\sigma}_A)\nu(\overline{\mathsf{Rol}}_q)|_q \\
& + (L_1|_q\alpha_2 - L_2|_q\alpha_1 + (\alpha_1-\beta_2)L_1|_q\phi + (\beta_1+\alpha_2)L_2|_q\phi - \beta_1\alpha_3 + \alpha_2\alpha_3)\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
& + (L_1|_q\beta_2 - L_2|_q\beta_1 + (\beta_1+\alpha_2)L_1|_q\phi + (-\alpha_1+\beta_2)L_2|_q\phi - \beta_3\beta_1 + \alpha_2\beta_3 \\
& \quad - (\alpha_1\beta_2 - \beta_1\alpha_2)\omega - \omega K)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

$$\begin{aligned}
& [L_2, L_3]|_q \\
= & (L_3|_q\phi - \alpha_2)L_1|_q - \beta_2 L_2|_q - \beta_3 L_3|_q + (\alpha_2\beta_3 - \beta_2\alpha_3 - \Pi_{\tilde{X}})\nu(\overline{\mathsf{Rol}}_q)|_q \\
& + (L_2|_q\alpha_3 - L_3|_q\alpha_2 - \beta_3 L_2|_q\phi + (-\alpha_1+\beta_2)L_3|_q\phi + \alpha_1\alpha_2 + \alpha_2\beta_2 + \alpha_3\beta_3 + \tilde{\Pi}_{\hat{Z}})\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
& + (L_2|_q\beta_3 - L_3|_q\beta_2 + \alpha_3 L_2|_q\phi + (-\alpha_2-\beta_1)L_3|_q\phi + \beta_1\alpha_2 + \beta_2^2 + \beta_3^2 \\
& \quad - (\alpha_2\beta_3 - \beta_2\alpha_3)\omega + \omega\Pi_{\tilde{X}} + \tilde{\hat{\sigma}}_A^1)\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q,
\end{aligned}$$

and,

$$
\begin{aligned}
&[L_3, L_1]|_q \\
=&\alpha_1 L_1|_q + (L_3|_q\phi + \beta_1)L_2|_q + \alpha_3 L_3|_q - (\alpha_1\beta_3 - \beta_1\alpha_3)\nu(\mathsf{Rol}_q)|_q \\
&+ (L_3|_q\alpha_1 - L_1|_q\alpha_3 - (\alpha_2+\beta_1)L_3|_q\phi + \beta_3 L_1|_q\phi - \alpha_1^2 - \alpha_2\beta_1 - \alpha_3^2 - \tilde{\tilde{\sigma}}_A^2)\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q \\
&+ (L_3|_q\beta_1 - L_1|_q\beta_3 + (\alpha_1-\beta_2)L_3|_q\phi - \alpha_3 L_1|_q\phi - \alpha_1\beta_1 - \beta_1\beta_2 - \alpha_3\beta_3 \\
&\quad + (\alpha_1\beta_3 - \beta_1\alpha_3)\omega - \tilde{\Pi}_{\hat{Z}})\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q.
\end{aligned}
$$

As we see before, we have the claim since the coefficients of $\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q$ and $\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q$ are equal to zero. □

Dim $\mathcal{O}_{\mathscr{D}_R}(q_0) = 7$: In this situation, we assume that $\dim\mathcal{O}_{\mathscr{D}_R}(q_0) = 7$. Then by using the vectors L_1, L_2 and L_3, one could vanish the coefficients of $\mathscr{L}_{NS}(A\tilde{X}_A)|_q$, $\mathscr{L}_{NS}(A\tilde{Y}_A)|_q$ and $\mathscr{L}_{NS}(\hat{Z}_A)|_q$ in the other vectors of the 8 column vectors, thus we obtain 5 vectors of the form

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi_i \\ \zeta_i \end{pmatrix}.
$$

where ξ_i, ζ_i, $i = 1,2,3,4,5$, are functions on Q. Since the dimension of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ is 7, then all these 5 vectors are linearly dependent to each other. Write one of these 5 vectors as $\xi_{i_0}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q + \zeta_{i_0}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q$ and then define the tangent vector fields W to $\mathcal{O}_{\mathscr{D}_R}(q_0)$

$$
\nu(\theta_W\otimes\hat{Z}_A)|_q := \xi_{i_0}\nu(\theta_{\tilde{X}_A}\otimes\hat{Z}_A)|_q + \zeta_{i_0}\nu(\theta_{\tilde{Y}_A}\otimes\hat{Z}_A)|_q.
$$

We set W^p as the orthogonal vector to W with respect to the frame X, Y. Thus, there exists an Q-dependent angle ψ such that

$$
\begin{aligned}
W &= cos\psi X + sin\psi Y, \\
W^p &= -sin\psi X + cos\psi Y.
\end{aligned}
$$

We denote $V|_q := \nu(\theta_W\otimes\hat{Z}_A)|_q$ and $V^p|_q := \nu(\theta_{W^p}\otimes\hat{Z}_A)|_q$. Since $V|_q$ is orthogonal to L_1, L_2, L_3 then we deduce the existence of the following three vectors,

$$
\begin{aligned}
G_1 &= \mathscr{L}_{NS}(AW)|_q + \lambda_X V^p|_q, \\
G_2 &= \mathscr{L}_{NS}(AW^p)|_q + \lambda_Y V^p|_q, \\
G_3 &= \mathscr{L}_{NS}(\hat{Z}_A)|_q + \lambda_{\hat{Z}} V^p|_q,
\end{aligned}
$$

where λ_X, λ_Y, $\lambda_{\hat{Z}}$ are functions on Q. In the new frame, $\overline{\mathsf{Rol}}$ is equal to

$$
\nu(\overline{\mathsf{Rol}}_q)|_q = \nu(A(X\wedge Y))|_q + aV|_q + bV^p|_q \tag{5.61}
$$

such that, a and b are equal to

$$
\begin{aligned}
a &= -\frac{\Pi_{W^p}}{K-\hat{\sigma}_A}, \\
b &= \frac{\Pi_W}{K-\hat{\sigma}_A}.
\end{aligned}
$$

Lemma 5.3.25. *We have the following derivatives on $\mathcal{O}_{\mathscr{D}_R}(q_0)$ with respect to $\mathscr{L}_R(W)|_q$,*

$$
\begin{cases}
\mathscr{L}_R(W)|_q\psi + g(\Gamma, W) = -\lambda_{\hat{Z}}, \\
\mathscr{L}_R(W)|_q\lambda_X = -\lambda_Y\lambda_{\hat{Z}}, \\
\mathscr{L}_R(W)|_q\lambda_Y = b\hat{\sigma}_A + \Pi_W + \lambda_X\lambda_{\hat{Z}}, \\
\mathscr{L}_R(W)|_q\lambda_{\hat{Z}} = -\Pi^p_{\hat{Z}} + b\Pi_{W^p}, \\
\mathscr{L}_R(W)|_q b = -\lambda_Y,
\end{cases}
$$

and with respect to $\mathscr{L}_R(W^p)|_q$,

$$
\begin{cases}
\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p) = 0, \\
\mathscr{L}_R(W^p)|_q\lambda_X = -\lambda_X\lambda_{\hat{Z}} - \Pi_W - b\hat{\sigma}_A, \\
\mathscr{L}_R(W^p)|_q\lambda_Y = -\lambda_Y\lambda_{\hat{Z}}, \\
\mathscr{L}_R(W^p)|_q\lambda_{\hat{Z}} = -(\hat{\sigma}^1_A)^p - b\Pi_W - \lambda^2_{\hat{Z}}, \\
\mathscr{L}_R(W^p)|_q b = \lambda_X - b\lambda_{\hat{Z}}.
\end{cases}
$$

Proof. Computing the Lie brackets on $\mathcal{O}_{\mathscr{D}_R}(q_0)$ with $\mathscr{L}_R(W)|_q$,

$$
\begin{aligned}
& [\mathscr{L}_R(W), G_1]|_q \\
= \; & -(G_1|_q\psi)\mathscr{L}_R(W^p)|_q \\
& +(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W))G_2|_q \\
& -\lambda_X(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W))V|_q \\
& +(\mathscr{L}_R(W)|_q\lambda_X - \lambda_Y(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W)))V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W), G_2]|_q \\
= \; & -(G_2|_q\psi)\mathscr{L}_R(W^p)|_q + \hat{\sigma}_A\nu(\overline{\mathsf{Rol}_q})|_q \\
& -(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W))G_1|_q \\
& +(-\lambda_Y(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W)) - a\hat{\sigma}_A + \Pi_{W^p})V|_q \\
& +(\mathscr{L}_R(W)|_q\lambda_Y - b\hat{\sigma}_A - \Pi_W + \lambda_X(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W)))V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W), G_3]|_q \\
= \; & -(G_3|_q\psi)\mathscr{L}_R(W^p)|_q + \Pi_{W^p}\nu(\overline{\mathsf{Rol}_q})|_q \\
& +(-\lambda_{\hat{Z}}(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W)) + (\hat{\sigma}^2_A)^p - a\Pi_{W^p})V|_q \\
& +(\mathscr{L}_R(W)|_q\lambda_{\hat{Z}} + \Pi^p_{\hat{Z}} - b\Pi_{W^p})V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W), V]|_q \\
= \; & -(V|_q\psi)\mathscr{L}_R(W^p)|_q \\
& -G_3|_q \\
& +(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W) + \lambda_{\hat{Z}})V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W), \nu(\overline{\mathsf{Rol}}_{(\cdot)})]|_q \\
= \; & -(\nu(\overline{\mathsf{Rol}}_q)|_q\psi)\mathscr{L}_R(W^p)|_q \\
& -G_2|_q - aG_3|_q \\
& +(\mathscr{L}_R(W)|_q a - b(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W)))V|_q \\
& +(\mathscr{L}_R(W)|_q b + a(\mathscr{L}_R(W)|_q\psi + g(\Gamma, W)) + \lambda_Y + a\lambda_{\hat{Z}})V^p|_q,
\end{aligned}
$$

and with $\mathscr{L}_R(W^p)|_q$,

$$
\begin{aligned}
& [\mathscr{L}_R(W^p), G_1]|_q \\
= \; & (G_1|_q\psi)\mathscr{L}_R(W)|_q - \hat{\sigma}_A\nu(\overline{\mathsf{Rol}}_q)|_q \\
& +(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p))G_2|_q - \lambda_X G_3|_q \\
& +(-\lambda_X(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p)) - \Pi_{W^p} + a\hat{\sigma}_A)V|_q \\
& +(\mathscr{L}_R(W^p)|_q\lambda_X - \lambda_Y(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p)) + \lambda_X\lambda_{\hat{Z}} + \Pi_W + b\hat{\sigma}_A)V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W^p), G_2]|_q \\
= \; & (G_2|_q\psi)\mathscr{L}_R(W)|_q \\
& -(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p))G_1|_q - \lambda_Y G_3|_q \\
& -\lambda_Y(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p))V|_q \\
& +(\mathscr{L}_R(W^p)|_q\lambda_Y + \lambda_X(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p)) + \lambda_Y\lambda_{\hat{Z}})V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W^p), G_3]|_q \\
= \; & (G_3|_q\psi)\mathscr{L}_R(W)|_q - \Pi_W\nu(\overline{\mathsf{Rol}}_q)|_q \\
& -\lambda_{\hat{Z}}G_3 \\
& +(-\lambda_{\hat{Z}}(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p)) + \Pi_{\hat{Z}}^p + a\Pi_W)V|_q \\
& +(\mathscr{L}_R(W^p)|_q\lambda_{\hat{Z}} + (\hat{\sigma}_A^1)^p + b\Pi_W + \lambda_{\hat{Z}}^2)V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W^p), V]|_q \\
= \; & (V|_q\psi)\mathscr{L}_R(W)|_q \\
& +(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p))V^p|_q,
\end{aligned}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(W^p), \nu(\overline{\mathsf{Rol}}_{(\cdot)})]|_q \\
= \; & (\nu(\overline{\mathsf{Rol}}_q)|_q\psi)\mathscr{L}_R(W)|_q \\
& +G_1|_q - bG_3|_q \\
& +(\mathscr{L}_R(W^p)|_q a - b(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p)))V|_q \\
& +(\mathscr{L}_R(W^p)|_q b + a(\mathscr{L}_R(W^p)|_q\psi + g(\Gamma, W^p)) - \lambda_X + b\lambda_{\hat{Z}})V^p|_q.
\end{aligned}
$$

Because we are in the case where $\dim \mathcal{O}_{\mathscr{D}_R}(q_0) = 7$, $V^p|_q$ cannot be tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Vanishing the coefficients of V^p in the above brackets, we get the claim.

5.4 Appendix

Definition 5.4.1. *On an oriented Riemannian manifold (M, g), one defines the Hodge-dual $\star_M$ as the linear map uniquely defined by*

$$\star_M : \wedge^k T_x M \to \wedge^{n-k} T_x M; \quad \star_M (X_1 \wedge ... \wedge X_k) = X_{k+1} \wedge ... \wedge X_n,$$

with $x \in M$, $k = 0, ..., n = dim M$ and $X_1 ... X_n \in T_x M$ any oriented basis.

For an oriented Riemannian manifold (M, g) and $x \in M$, if A, $B \in \mathfrak{so}(T_x M)$, we define

$$[A, B]_{\mathfrak{so}} := A \circ b - B \circ A \in \mathfrak{so}(T_x M).$$

Also, we define the following natural isomorphism ϕ by

$$\phi : \wedge^2 TM \to \mathfrak{so}(M); \quad \phi(X \wedge Y) := g(., X)Y - g(., Y)X.$$

Using this isomorphism, we may consider, for each $x \in M$, the curvature tensor R of (M, g) as a linear map,

$$\mathcal{R} : \wedge^2 T_x M \to \wedge^2 T_x M; \quad \mathcal{R}(X \wedge Y) := \phi^{-1}(R(X, Y)),$$

where X, $Y \in T_x M$. Here of course $R(X, Y)$, as an element of $T_x^* M \otimes T_{\hat{x}} \hat{M}$, belongs to $\mathfrak{so}(T_x M)$.
It is standard fact that $\mathcal{R}$ is a symmetric map when $\wedge^2(T_x M)$ is endowed with the inner product, also written as g,

$$g(X \wedge Y, Z \wedge W) := g(X, Z)g(Y, Z) - g(X, W)g(Y, Z).$$

Notice also that for A, $B \in \mathfrak{so}(T_x M)$,

$$tr(AB) = g(\phi^{-1}(A), \phi^{-1}(B)).$$

The map $\mathcal{R}$ is usually called the curvature operator and we will, with a slight abuse of notation, write it simply as R.
In dimension $n = 3$, one has $\star_M^2 = id$ when $\star_M$ is the map $\wedge^2 TM \to TM$ and $TM \to \wedge^2 TM$. Let X, Y, $Z \in T_x M$ be an orthonormal positively oriented basis. Then

$$\star_M (X \wedge Y) = Z, \quad \star_M (Y \wedge Z) = X, \quad \star_M (Z \wedge x) = Y.$$

In other hand, we have

$$\phi(\star_M \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}) = \phi(\alpha(\star_M X) + \beta(\star_M Y) + \gamma(\star_M Z)) = \phi(\alpha Y \wedge Z + \beta Z \wedge X + \gamma X \wedge Y)$$

$$= \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix}.$$

Then, in terms of this basis X, Y, Z one has

$$\star_M \phi^{-1}\begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

And one has

$$[\phi(\star X), \phi(\star Y)]_{\mathfrak{so}} = \phi(X \wedge Y).$$

Now, in dimension $n = 2$, one has $-\star_M^2 = id$ when $\star_M$ is the map $TM \to TM$. Recall that for any orthonormal frame E_1, E_2, E_3 of 3-dimensional Riemmannian manifold, the second Bianchi identity has the form

$$(\nabla_{E_1} R)(\star E_1) + (\nabla_{E_2} R)(\star E_2) + (\nabla_{E_3} R)(\star E_3) = 0. \tag{5.62}$$

We make also the following technical definition. Recall that, if $E_1, ..., E_n$ is an g-orthonormal frame of a n-dimensional Riemannian manifold (M, g), the Koszul formula simplifies to

$$2\Gamma^i_{j,k} := 2g(\nabla_{F_i} F_j, F_k) = g([F_i, F_j], F_k) - g([F_i, F_k], F_j) - g([F_j, F_k], F_i).$$

Since $E_1, ..., E_n$ is orthonormal, one has $\Gamma^i_{j,k} = -\Gamma^i_{k,j}$ for all i, j, k. This relation mean that in the case where the dimension $n = 3$, we get the matrix

$$\Gamma = \begin{pmatrix} \Gamma^1_{(2,3)} & \Gamma^2_{(2,3)} & \Gamma^3_{(2,3)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & \Gamma^2_{(1,2)} & \Gamma^3_{(1,2)} \end{pmatrix},$$

represent the connection table with respect to E_1, E_2, E_3. Similarly, if $n = 2$, the connection has the form of the following matrix

$$\begin{pmatrix} \Gamma^1_{(1,2)} & \Gamma^2_{(1,2)} \end{pmatrix}.$$

Furthermore, suppose that we have a 3-dimensional Riemannian manifold $\hat{M}$ of Riemannian curvature tensor $\hat{R}$. One can write, with respect to a local frame of $\hat{M}$ $\hat{E}_1, \hat{E}_2, \hat{E}_3$, the coefficients of $\hat{R}$ in terms of the coefficients of the connection $\hat{\Gamma}$ of $\hat{M}$. So, if

$$\hat{\Gamma} = \begin{pmatrix} \hat{\Gamma}^1_{(2,3)} & \hat{\Gamma}^2_{(2,3)} & \hat{\Gamma}^3_{(2,3)} \\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & \hat{\Gamma}^2_{(1,2)} & \hat{\Gamma}^3_{(1,2)} \end{pmatrix},$$

and if

$$\hat{R}(\hat{E}_1, \hat{E}_2) = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \hat{R}(\hat{E}_2, \hat{E}_3) = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \hat{R}(\hat{E}_3, \hat{E}_1) = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix},$$

then we have

$$
\begin{aligned}
a_1 &= \hat{E}_1(\hat{\Gamma}^2_{(2,3)}) - \hat{E}_2(\hat{\Gamma}^1_{(2,3)}) + (\hat{\Gamma}^1_{(3,1)} + \hat{\Gamma}^2_{(2,3)})\hat{\Gamma}^2_{(1,2)} \\
&+(-\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^1_{(2,3)})\hat{\Gamma}^3_{(2,3)} + (\hat{\Gamma}^1_{(2,3)} - \hat{\Gamma}^2_{(3,1)})\hat{\Gamma}^1_{(1,2)}, \\
b_1 &= \hat{E}_1(\hat{\Gamma}^2_{(3,1)}) - \hat{E}_2(\hat{\Gamma}^1_{(3,1)}) + (\hat{\Gamma}^1_{(3,1)} + \hat{\Gamma}^2_{(2,3)})\hat{\Gamma}^1_{(1,2)} \\
&+(-\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^1_{(2,3)})\hat{\Gamma}^3_{(3,1)} + (\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^1_{(2,3)})\hat{\Gamma}^2_{(1,2)}, \\
c_1 &= \hat{E}_1(\hat{\Gamma}^2_{(1,2)}) - \hat{E}_2(\hat{\Gamma}^1_{(1,2)}) + (\hat{\Gamma}^1_{(1,2)})^2 + (-\hat{\Gamma}^1_{(2,3)} - \hat{\Gamma}^2_{(3,1)})\hat{\Gamma}^3_{(1,2)} \\
&+\hat{\Gamma}^1_{(2,3)}\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^1_{(3,1)}\hat{\Gamma}^2_{(2,3)} + (\hat{\Gamma}^2_{(1,2)})^2, \\
a_2 &= \hat{E}_2(\hat{\Gamma}^3_{(2,3)}) - \hat{E}_3(\hat{\Gamma}^2_{(2,3)}) + (\hat{\Gamma}^2_{(2,3)})^2 + (-\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^3_{(1,2)})\hat{\Gamma}^1_{(2,3)} \\
&+\hat{\Gamma}^2_{(3,1)}\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^2_{(1,2)}\hat{\Gamma}^3_{(3,1)} + (\hat{\Gamma}^3_{(2,3)})^2, \\
b_2 &= \hat{E}_2(\hat{\Gamma}^3_{(3,1)}) - \hat{E}_3(\hat{\Gamma}^2_{(3,1)}) + (\hat{\Gamma}^3_{(3,1)} + \hat{\Gamma}^2_{(1,2)})\hat{\Gamma}^3_{(2,3)} \\
&+(-\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^3_{(1,2)})\hat{\Gamma}^1_{(3,1)} + (\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^3_{(1,2)})\hat{\Gamma}^2_{(2,3)}, \\
c_2 &= \hat{E}_2(\hat{\Gamma}^3_{(1,2)}) - \hat{E}_3(\hat{\Gamma}^2_{(1,2)}) + (\hat{\Gamma}^3_{(3,1)} + \hat{\Gamma}^2_{(1,2)})\hat{\Gamma}^2_{(2,3)} \\
&+(-\hat{\Gamma}^2_{(3,1)} - \hat{\Gamma}^3_{(1,2)})\hat{\Gamma}^1_{(1,2)} + (\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^2_{(3,1)})\hat{\Gamma}^3_{(2,3)}, \\
a_3 &= \hat{E}_3(\hat{\Gamma}^1_{(2,3)}) - \hat{E}_1(\hat{\Gamma}^3_{(2,3)}) + (\hat{\Gamma}^3_{(2,3)} + \hat{\Gamma}^1_{(1,2)})\hat{\Gamma}^3_{(3,1)} \\
&+(-\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^1_{(2,3)})\hat{\Gamma}^2_{(2,3)} + (\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^3_{(1,2)})\hat{\Gamma}^1_{(3,1)}, \\
b_3 &= \hat{E}_3(\hat{\Gamma}^1_{(3,1)}) - \hat{E}_1(\hat{\Gamma}^3_{(3,1)}) + (\hat{\Gamma}^1_{(3,1)})^2 + (-\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^1_{(2,3)})\hat{\Gamma}^2_{(3,1)} \\
&+\hat{\Gamma}^1_{(2,3)}\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^3_{(2,3)}\hat{\Gamma}^1_{(1,2)} + (\hat{\Gamma}^3_{(3,1)})^2, \\
c_3 &= \hat{E}_3(\hat{\Gamma}^1_{(1,2)}) - \hat{E}_1(\hat{\Gamma}^3_{(1,2)}) + (\hat{\Gamma}^1_{(1,2)} + \hat{\Gamma}^3_{(2,3)})\hat{\Gamma}^1_{(3,1)} \\
&+(-\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^1_{(2,3)})\hat{\Gamma}^2_{(3,1)} + (\hat{\Gamma}^3_{(1,2)} - \hat{\Gamma}^1_{(2,3)})\hat{\Gamma}^3_{(3,1)}.
\end{aligned}
\tag{5.63}
$$

Definition 5.4.2. *Let (M,g), (N,h) be Riemannian manifolds and $f \in C^\infty(M)$. Define a metric h_f on $M \times N$*

$$h_f = pr_1^*(g) + (f \circ pr_1)^2 pr_2^*(h),$$

where pr_1, pr_2 are projections onto the first and second factor of $M \times N$, respectively. Then, the Riemannian manifold $(M \times N, h_f)$ is called a warped product of (M,g) and (N,h) with the warping function f.

Definition 5.4.3. *A 3-dimensional Riemannian manifold (M,g) is said to belong to class $\mathcal{M}_\beta$, for $\beta \in \mathbb{R}$, if there exists an orthonormal frame $E_1, E_2, E_3 \in VF(M)$ with respect to which the connection table is of the form*

$$\Gamma = \begin{pmatrix} \beta & 0 & 0 \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ 0 & 0 & \beta \end{pmatrix},$$

in this case the frame E_1, E_2, E_3 is called an adapted frame of (M, g).

We have the following theorem.

Theorem 5.4.4. *Let (M, g) be a Riemannian manifold of dimension 3. Suppose that at every point $x_0 \in M$ there is an orthonormal frame E_1, E_2, E_3 defined in a neighbourhood of x_0 such that the connection table with respect to E_1, E_2, E_3 on this neighbourhood is of the form*

$$\Gamma = \begin{pmatrix} 0 & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & 0 \end{pmatrix},$$

and moreover $X(\Gamma^1_{(1,2)}) = 0$, $\forall X \in E_2^\perp$.

Then there is a neighbourhood U of x, an interval $I \subset \mathbb{R}$, $f \in C^\infty(I)$ and a 2-dimensional Riemannian manifold (N, h) such that $(U, g|_U)$ is isometric to the warped product $(I \times N, h_f)$. If F: $(I \times N, h_f) \to (U, g|_U)$ is this isometry, then for all $(r, y) \in I \times N$,

$$\frac{f'(r)}{f(r)} = -\Gamma^1_{(1,2)}(F(r, y)), \tag{5.64}$$

$$F_* \frac{\partial}{\partial r}|_{(r,y)} = E_2|_{F(r,y)}. \tag{5.65}$$

In the previous theorem, we write $\frac{\partial}{\partial r}$ for the natural positively directed unit vector field on $\mathbb{R}$ with respect to the standard Euclidean metric, we identity it in the canonical way as a vector field on the product $I \times N$ and notice that it is also a unit vector field with respect to h_f.

Chapter 6

Horizontal Holonomy for Affine Manifolds

In this chapter, we consider a smooth connected finite-dimensional manifold M, a complete affine connection ∇ with holonomy group H^{∇} and a smooth completely non integrable distribution Δ. We define the Δ-horizontal holonomy group H_{Δ}^{∇} as the subgroup of H^{∇} obtained by ∇-parallel transporting frames only along loops tangent to Δ. We first set elementary properties of H_{Δ}^{∇} and show how to study it using the development formalism (see Section 3.2 in Chapter 1). In particular, it is shown that H_{Δ}^{∇} is a Lie group. Moreover, we study an explicit example where M is a free step-two homogeneous Carnot group and ∇ is the Levi-Civita connection associated to a Riemannian metric on M, and show that in this particular case H_{Δ}^{∇} is compact and strictly included in H^{∇}.

6.1 Affine Holonomy Group of (M, ∇, Δ)

6.1.1 Definitions

Consider the triple (M, ∇, Δ) where M is a smooth manifold, ∇ a affine connection on M and Δ a completely controllable smooth distribution on M. In this section, we will restrict Definition 2.2.1 to the Δ-admissible curves on M. To this end, we will define the set of all Δ-admissible loop based at points of M.

Definition 6.1.1. *We define $\Omega_{\Delta}(x)$ the set of all a.c. Δ-admissible loops based at x, as*

$$\Omega_{\Delta}(x) := \{\gamma \mid \gamma : [a,b] \to M \text{ a.c., } \gamma(a) = \gamma(b) = x \text{ and } \dot{\gamma}(t) \in \Delta\big|_{\gamma(t)} \text{ a.e.}\}$$

The following result is immediate from the definitions.

Proposition 6.1.2. *The set $\Omega_{\Delta}(x)$ of all a.c. Δ-admissible loop based at x is not empty and is closed under the operation " $\cdot$ " given in* (2.5).

We define the holonomy group associated with the distribution Δ as follows.

Definition 6.1.3. *For every $x \in M$, the holonomy group associated with Δ at x is defined as*

$$H_{\Delta}^{\nabla}|_x := \{(P^{\nabla})_0^1(\gamma) \mid \gamma \in \Omega_{\Delta}(x)\}.$$

Proposition 6.1.4. *For every $x, y \in M$, $H_{\Delta}^{\nabla}|_x$ is a subgroup of $H^{\nabla}|_x$ and $H_{\Delta}^{\nabla}|_x$ is conjugate to $H_{\Delta}^{\nabla}|_y$. One can thus define $H_{\Delta}^{\nabla} \subset H^{\nabla} \subset \mathrm{GL}(n)$ and we call it the Δ-horizontal holonomy group associated with Δ and the affine connection ∇.*

Proof. Since $\Omega_{\Delta}(x)$ is a nonempty set for any $x \in M$, then $H_{\Delta}^{\nabla}|_x$ is also a nonempty subset of $H^{\nabla}|_x$. By Definitions 2.2.1 and 2.2.2 of [23], the inverse map of $(P^{\nabla})_0^1(\gamma)$ is $(P^{\nabla})_0^1(\gamma^{-1})$ and $(P^{\nabla})_0^1(\delta) \circ (P^{\nabla})_0^1(\gamma)$ is equal to $(P^{\nabla})_0^1(\delta \cdot \gamma)$, for any $\gamma : [0, 1] \to M$ and $\delta : [0, 1] \to M$ belonging to $\Omega_{\Delta}(x)$. Thus, we get the first statement. Next, taking into account the fact that Δ is completely controllable, one deduces the rest of the proposition.

Remark 6.1.5. If g is a Riemannian metric on the smooth manifold M and ∇^g is the Levi-Civita connection associated to g, then the holonomy group $H^{\nabla^g}|_x$ with $x \in M$ is a subgroup of $O(T_xM)$, the set of g-orthogonal transformations of T_xM. If, moreover, M is oriented, one can easily prove that $H^{\nabla^g}|_x$ is a subgroup of $\mathrm{SO}(T_xM)$. One can then define the holonomy group of ∇^g as a subgroup of $O(n)$ ($SO(n)$ respectively) the group of orthogonal transformations of the euclidean n-dimensional space (the subgroup of $O(n)$ with determinant equal to one if M is oriented respectively). We of course get similar statements for $H_{\Delta}^{\nabla^g}$. Note that $(H^{\nabla^g})^0$ and $(H_{\Delta}^{\nabla^g})^0$ are compact subgroups of $O(n)$ ($SO(n)$ respectively) according to Weyl's theorem (cf. [30, Theorem 26.1]) since any Lie-subalgebra of a compact Lie-algebra is compact.

6.1.2 Holonomy groups associated with distributions using the framework of rolling manifolds

Let M be a smooth n-dimensional manifold and ∇ a connection on M. Set $(\hat{M}, \hat{\nabla}) := (\mathbb{R}^n, \hat{\nabla}^n)$ where $\hat{\nabla}^n$ is the Euclidean connection on $\mathbb{R}^n$. We associate to (M, ∇) the curvature tensor R^{∇} and to the product manifold $(M, \nabla) \times (\mathbb{R}^n, \hat{\nabla}^n)$ the affine connection $\overline{\nabla}$.

Affine Holonomy Group of M

One can extend readily Proposition 3.10 of [12] to get the following result.

Proposition 6.1.6. *For any $f \in \mathrm{Aff}(M)$, $\hat{f} \in \mathrm{Aff}(n)$ and any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, define the following smooth right and left actions of* $\mathrm{Aff}(M)$ *and* $\mathrm{Aff}(n)$ *on Q*

$$q_0 \cdot f := (f^{-1}(x_0), \hat{x}_0; A_0 \circ f_{\star}|_{f^{-1}(x_0)}), \qquad \hat{f} \cdot q_0 := (x_0, \hat{f}(\hat{x}_0); \hat{f}_{\star}|_{\hat{x}_0} \circ A_0).$$

Then, for any a.c. curve $\gamma : [0, 1] \to M$ starting at x_0, one has for a.e. $t \in [0, 1]$

$$\hat{f} \cdot q_{\mathscr{D}_R}(\gamma, q_0)(t) \cdot f = q_{\mathscr{D}_R}(f^{-1} \circ \gamma, \hat{f} \cdot q_0 \cdot f)(t).$$

Proof. By the definition of an affine transformation f on M, we have Eq. (2.3) for any a.c. curve $\gamma : [0,1] \to M$. This implies that, for a.e. $t \in [0,1]$

$$f_\star|_{\gamma(t)} \circ (P^\nabla)_0^t(\gamma) = (P^\nabla)_0^t(f \circ \gamma) \circ f_\star|_{\gamma(0)}.$$

We have the same conclusion for affine transformations $\hat{f}$ on $\mathbb{R}^n$. Then, since $\mathrm{Aff}(n)$ is a Lie group and by what precedes, one can repeat the steps of the proof of Proposition 3.10 in [12] with the group $\mathrm{Aff}(n)$ instead of isometry groups on M and $\mathbb{R}^n$ to get the claim.

Recall that if G is a Lie group, then a smooth bundle $\pi : E \to M$ is a principal G-bundle over M if there exists a smooth and free action of G on E which preserves the fibers of π, cf. [23]. Furthermore, we recall that the affine group $\mathrm{Aff}(n)$ is equal to $\mathbb{R}^n \rtimes GL(n)$ and its product group $\diamond$ is given by

$$(v, L) \diamond (u, K) := (Lu + v, L \circ K).$$

Using the previous proposition, one can extend immediately the simple but crucial Proposition 4.1 in [12] to derive the next result.

Proposition 6.1.7. *The bundle $\pi_{Q,M} : Q \to M$ is a principal $\mathrm{Aff}(n)$-bundle with the left action $\mu : \mathrm{Aff}(n) \times Q \to Q$;*

$$\mu((\hat{y}, C), (x, \hat{x}; A)) = (x, C\hat{x} + \hat{y}; C \circ A).$$

The action μ preserves $\mathscr{D}_R$, i.e. for any $q \in Q$ and $B \in \mathrm{Aff}(n)$, we have $(\mu_B)_*\mathscr{D}_R|_q = \mathscr{D}_R|_{\mu_B(q)}$ where $\mu_B : Q \to Q$; $q \mapsto \mu(B, q)$. Moreover, for any $q = (x, \hat{x}; A) \in Q$, there exists a unique subgroup $\mathcal{H}_q^\nabla$ of $\mathrm{Aff}(n)$, called the affine holonomy group of (M, ∇) verifying

$$\mu(\mathcal{H}_q^\nabla \times \{q\}) = \mathcal{O}_{\mathscr{D}_R}(q) \cap \pi_{Q,M}^{-1}(x).$$

If $q' = (x, \hat{x}'; A') \in Q$ belongs to the same $\pi_{Q,M}$-fiber as q, then $\mathcal{H}_q^\nabla$ and $\mathcal{H}_{q'}^\nabla$ are conjugate in $\mathrm{Aff}(n)$ and all conjugacy classes of $\mathcal{H}_q^\nabla$ are of the form $\mathcal{H}_{q'}^\nabla$. This conjugacy class is denoted by $\mathcal{H}^\nabla$ and its projection of $GL(n)$ is equal to H^∇ the holonomy group of the affine connection ∇.

Proof. Let $q = (x, \hat{x}; A) \in Q$ and $B = (\hat{y}, C) \in \mathrm{Aff}(n)$. Since $C \circ A$ is in $\mathrm{GL}(n)$, then $\mu(B, q) \in Q$. In order to prove that μ is transitive and proper, we can follow the same steps of the proof of Proposition 4.1 in [12] due to Proposition 6.1.6.

Affine Holonomy Group on Δ

Consider now a smooth completely controllable distribution Δ on (M, ∇). We will determine the sub-distribution of $\mathscr{D}_R$ by restriction to Δ instead of considering the whole tangent space of M.

Definition 6.1.8. *The rolling distribution Δ_R on Δ is the smooth sub-distribution of $\mathscr{D}_R$ defined on $(x, \hat{x}; A) \in Q$ by*

$$\Delta_R|_{(x,\hat{x};A)} = \mathscr{L}_R(\Delta|_x)|_{(x,\hat{x};A)}. \tag{6.1}$$

Since Δ is completely controllable, we use Proposition 3.2.6 to obtain the next corollary.

Corollary 6.1.9. *For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and any a.c. Δ-admissible curve $\gamma : [0,1] \to M$ starting at x_0, there exists a unique a.c. Δ_R-admissible curve $q_{\Delta_R}(\gamma, q_0) : [0,T] \to Q$ where $0 < T \leq 1$.*

Since we can easily restrict the proof of Proposition 6.1.7 (cf. [12]) on Δ_R, we get the next proposition.

Corollary 6.1.10. *The action μ mentioned in Proposition 6.1.7 preserves the distribution Δ_R. Moreover, for every $q \in Q$, there exists a unique* algebraic *subgroup $\mathcal{H}^{\nabla}_{\Delta_R|q}$ of $\mathcal{H}^{\nabla}_q$, called the affine holonomy group of Δ_R, such that*

$$\mu(\mathcal{H}^{\nabla}_{\Delta_R|q} \times \{q\}) = \mathcal{O}_{\Delta_R}(q) \cap \pi^{-1}_{Q,M}(x),$$

where $x = \pi_{Q,M}(q)$ and $\mathcal{O}_{\Delta_R}(q)$ is the Δ_R-orbit at q.

As before, one gets the following: if $q' = (x, \hat{x}'; A') \in Q$ belongs to the same $\pi_{Q,M}$-fiber as q, then $\mathcal{H}^{\nabla}_{\Delta_R|q}$ and $\mathcal{H}^{\nabla}_{\Delta_R|q'}$ are conjugate in $\mathrm{Aff}(n)$ and all conjugacy classes of $\mathcal{H}^{\nabla}_{\Delta_R|q}$ are of the form $\mathcal{H}^{\nabla}_{\Delta_R|q'}$. This conjugacy class is denoted by $\mathcal{H}^{\nabla}_{\Delta_R}$ and its projection of $GL(n)$ is a subgroup of H^{∇} which is equal to the Δ-horizontal holonomy group associated with Δ and the affine connection ∇.

Definition 6.1.11. *We denote by $\mathcal{O}^{loop}_{\Delta_R}(q_0)$ the set of the end points of the rolling development curves with initial conditions any point $q_0 = (x_0, \hat{x}_0; A_0)$ and any a.c. Δ-admissible loop at x_0, i.e., for $q_0 = (x_0, \hat{x}_0; A_0) \in Q$,*

$$\mathcal{O}^{loop}_{\Delta_R}(q_0) = \{q_{\Delta_R}(\gamma, q_0)(1) \mid \gamma : [0,1] \to M, \ \textit{a.c. } \Delta\textit{-admissible loop at } x_0\}.$$

If we fix a point q_0 of $Q = Q(M, \mathbb{R}^n)$ where the initial contact point on M is equal to x_0 and that on $\mathbb{R}^n$ is the origin, then we may consider the rolling development of M along a loop based at x_0. Then, one obtains a control problem whose state space is the fiber $\pi^{-1}_{Q,M}(x_0)$ and the reachable set is in the fiber $\pi^{-1}_{Q,M}(x_0)$ (for more details, cf. [12]). Then, $\mathcal{O}^{loop}_{\Delta_R}(q_0)$ is trivially in bijection with $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)$ and so $\mu(\mathcal{H}^{\nabla}_{\Delta_R|q_0} \times \{q_0\}) \simeq \mathcal{O}^{loop}_{\Delta_R}(q_0)$.

Proposition 6.1.12. *For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ the restriction of $\pi_{Q,M} : Q \to M$ into the orbit $\mathcal{O}_{\Delta_R}(q_0)$ is a submersion onto M.*

Proof. Clearly it is enough to show that $(\pi_{Q,M})_* T_{q_0} \mathcal{O}_{\Delta_R}(q_0) = T_{x_0} M$. Also recall that by the assumption of complete controllability of Δ we have $M = \mathcal{O}_{\Delta}(x_0)$.

Write $E^{x,t}(u)$ and $\tilde{E}^{q,t}(u)$ for the end-point maps of Δ and Δ_R starting from $x \in M$ and $q \in Q$, respectively. One easily sees that E and $\tilde{E}$ are related by

$$\pi_{Q,M} \circ \tilde{E}^{q,t} = E^{x,t}, \tag{6.2}$$

for any $q = (x, \hat{x}; A) \in Q$ and t where defined. We also denote by k the rank of Δ (i.e. the rank of Δ_R).

Let $\overline{u} \in L^2([0,1], \mathbb{R}^k)$ be any o-regular control of $E^{x_0,1}$ which belongs to the domain of definition of $\tilde{E}^{q_0,1}$. The existence of such an $\overline{u}$ is guaranteed by an application of Proposition 6.3.7 given in the appendix and Proposition 3.2.6, as in this case $(\hat{M}, \hat{g}) = \mathbb{R}^n$ is complete.

Let then $X \in T_{x_0}M$ be arbitrary, and notice that $T_{x_0}\mathcal{O}_\Delta(x_0) = T_{x_0}M$. By o-regularity of $\overline{u}$ with respect to $E^{x_0,1}$, there exists a C^1-map $u : I \to L^2([0,1], \mathbb{R}^k)$, where I is an open neighbourhood of 0, such that $u(0) = \overline{u}$ and $h(t,s) := E^{x_0,t}(u(s))$, $(t,s) \in [0,1] \times I$, satisfy $\frac{\partial}{\partial s} h(1,s)|_{s=0} = X$. Indeed, let $G : I \to \mathcal{O}_\Delta(x_0)$ be any smooth curve such that $\dot{G}(0) = X$. The o-regularity of $\overline{u}$ means that $D_u E^{x_0,1}$, i.e. the differential of $E^{x_0,1}$ at u, is surjective linear map from $L^2([0,1], \mathbb{R}^k)$ onto $T_{E^{x_0,1}(u)}\mathcal{O}_\Delta(x_0)$ when $u = \overline{u}$, and hence for all u close to $\bar{u}$ in $L^2([0,1], \mathbb{R}^k)$. One next defines $P(u)$ as the Moore-Penrose inverse of $D_u E^{x_0,1}$ and one considers the Cauchy problem $\frac{du(s)}{ds} = P(u(s))\frac{dG(s)}{ds}$, $u(0) = \bar{u}$. Then [7, Proposition 2] asserts that the maximal solution $u(\cdot)$ of the Cauchy problem is well-defined on a non empty interval centered at zero, which concludes the argument of the claim (after shrinking I if necessary).

Write $\tilde{h}(t,s) = \tilde{E}^{q_0,t}(u(s))$ for $(t,s) \in [0,1] \times I$. For each $i = 1, \dots, d$ and $s \in I$, the maps $t \mapsto h(t,s)$ and $t \mapsto \tilde{h}(t,s)$ are absolutely continuous and Δ- and Δ_R-admissible curves, respectively, and $h(t,s) = \pi_{Q,M}(\tilde{h}(t,s))$ by (6.2). In particular, $\frac{\partial}{\partial s}\tilde{h}(1,s)|_{s=0}$ is a vector in $T_{q_0}\mathcal{O}_{\Delta_R}(q_0)$ and

$$(\pi_{Q,M})_*\big(\frac{\partial}{\partial s}\tilde{h}(1,s)|_{s=0}\big) = \frac{\partial}{\partial s} h(1,s)|_{s=0} = X,$$

which shows that $X \in (\pi_{Q,M})_*(T_{q_0}\mathcal{O}_{\Delta_R}(q_0))$. Because X was arbitrary tangent vector of M at x_0, we conclude that $T_{x_0}M \subset (\pi_{Q,M})_*(T_{q_0}\mathcal{O}_{\Delta_R}(q_0))$.

The opposite inclusion $(\pi_{Q,M})_*(T_{q_0}\mathcal{O}_{\Delta_R}(q_0)) \subset T_{x_0}M$ being trivially true, this completes the proof.

Remark 6.1.13. Here is an alternative proof in the case that the distribution Δ satisfies LARC on a connected manifold M i.e. $Lie_x(\Delta) = T_xM$ for all $x \in M$.

Given vector fields $Y_1, \dots, Y_r$ and a subset $J = \{i_1, \dots, i_l\}$ of $\{1, \dots, r\}$ we write Y_J for the iterated bracket $[Y_{i_1}, [Y_{i_2}, \dots [Y_{i_{l-1}}, Y_{i_l}] \dots]$ of length l. Given $X \in T_{x_0}M = T_{x_0}\mathcal{O}_\Delta(x_0)$, there are, by the assumption, vector fields $Y_1, \dots, Y_r$ tangent to Δ, subsets $J_1, \dots, J_t$ of $\{1, \dots, r\}$ and numbers $a_1, \dots, a_t$ such that $X = \sum_{s=1}^t a_s Y_{J_s}|_{x_0}$. The lifts $\mathscr{L}_R(Y_i)$, $i = 1, \dots, r$ are tangent to Δ_R and satisfy $(\pi_{Q,M})_*\mathscr{L}_R(Y_i) = Y_i$, hence if we write $\mathscr{L}_R(Y)_J$ for $[\mathscr{L}_R(Y_{i_1}), [\mathscr{L}_R(Y_{i_2}), \dots [\mathscr{L}_R(Y_{i_{l-1}}), \mathscr{L}_R(Y_{i_l})] \dots]$ when J is as above, we have that $\mathscr{L}_R(Y)_{J_s}$ is tangent to $\mathcal{O}_{\Delta_R}(q_0)$ for every $s=1,\dots,t$

$$(\pi_{Q,M})_*|_{q_0}\big(\sum_{s=1}^{t} a_s\mathscr{L}_R(Y)_{J_s}\big)=\sum_{s=1}^{t} a_s Y_{J_s}|_{x_0}=X,$$

i.e. $X\in(\pi_{Q,M})_*T_{q_0}\mathcal{O}_{\Delta_R}(q_0)$. By arbitrariness of X in $T_{x_0}M$ we have the claimed submersivity of $\pi_{Q,M}$.

Classical results now apply to give the following.

Corollary 6.1.14. *In particular, for any $x\in M$ the fiber $\pi_{Q,M}^{-1}(x)\cap\mathcal{O}_{\Delta_R}(q_0)$ of $\mathcal{O}_{\Delta_R}(q_0)$ over x is either empty or a (closed) embedded submanifold of $\mathcal{O}_{\Delta_R}(q_0)$ of dimension $\delta=\dim\mathcal{O}_{\Delta_R}(q_0)-\dim M$.*

We arrive at the main result of this subsection.

Proposition 6.1.15. *Assume that Δ is a constant rank completely controllable distribution on (M,∇) where M is a connected smooth manifold and ∇ an affine connection. Then, the Δ-horizontal holonomy group H^{∇}_{Δ} and the affine holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ of Δ_R as defined previously are Lie subgroups of* $\mathrm{Aff}(n)$.

Proof. It is enough to prove the claim for $\mathcal{H}^{\nabla}_{\Delta_R}$. We first argue that $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ is an algebraic subgroup of $\mathrm{Aff}(n)$. To this end, to any $p\in\pi_{Q,M}^{-1}(x_0)$ (i.e. p is an arbitrary element of the fiber of Q over x_0) we match a unique $(y_p,C_p)\in\mathrm{Aff}(n)$ such that $\mu((y_p,C_p),q_0)=p$. Recall that $\mathcal{O}_{\Delta_R}(q_0)\cap\pi_{Q,M}^{-1}(x_0)$ is identified with $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ through this correspondence.

Then given $p_1,p_2\in\mathcal{O}_{\Delta_R}(q_0)\cap\pi_{Q,M}^{-1}(x_0)$, there are Δ-admissible (piecewise smooth) loops $\gamma_1,\gamma_2\in\Omega_M(x_0)$ in M based at x_0 such that $p_i=q_{\Delta_R}(\gamma_i,q_0)(1)$ for $i=1,2$. Letting $p=q_{\Delta_R}(\gamma_1\cdot\gamma_2,q_0)(1)$ we have

$$\begin{aligned}\mu((y_p,C_p),q_0)=&p=q_{\Delta_R}(\gamma_1\cdot\gamma_2,q_0)(1)=q_{\Delta_R}\big(\gamma_1,q_{\Delta_R}(\gamma_2,q_0)\big)(1)=q_{\Delta_R}\big(\gamma_1,p_2\big)(1)\\ =&q_{\Delta_R}\big(\gamma_1,\mu((y_{p_2},C_{p_2}),q_0)\big)(1)=\mu((y_{p_2},C_{p_2}),q_{\Delta_R}\big(\gamma_1,q_0)(1)\big)\\ =&\mu((y_{p_2},C_{p_2}),p_1)=\mu\big((y_{p_2},C_{p_2}),\mu((y_{p_1},C_{p_1}),q_0)\big)\\ =&\mu\big((y_{p_2},C_{p_2})(y_{p_1},C_{p_1}),q_0\big),\end{aligned}$$

i.e. $(y_p,C_p)=(y_{p_2},C_{p_2})(y_{p_1},C_{p_1})$, because the action μ is free. Since $\gamma_1\cdot\gamma_2$ is Δ-admissible loop, we have $p=q_{\Delta_R}(\gamma_1\cdot\gamma_2,q_0)(1)\in\mathcal{O}_{\Delta_R}(q_0)\cap\pi_{Q,M}^{-1}(x_0)$ i.e. $(y_p,C_p)\in\mathcal{H}^{\nabla}_{\Delta_R|q_0}$, and therefore $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ is indeed an algebraic subgroup of $\mathrm{Aff}(n)$ as claimed.

In other words we have shown that if $m:\mathrm{Aff}(n)\times\mathrm{Aff}(n)\to\mathrm{Aff}(n)$ is the smooth group multiplication operation on $\mathrm{Aff}(n)$, then

$$m(\mathcal{H}^{\nabla}_{\Delta_R|q_0}\times\mathcal{H}^{\nabla}_{\Delta_R|q_0})\subset\mathcal{H}^{\nabla}_{\Delta_R|q_0}.$$

By the orbit theorem 6.3.2 as given in the appendix (see also [17]), we know that any smooth map $f:Z\to Q$ for any smooth manifold Z such that $f(Z)\subset\mathcal{O}_{\Delta_R}(q_0)$ is smooth as a map $f : Z \to \mathcal{O}_{\Delta_R}(q_0)$. In other words, $\mathcal{O}_{\Delta_R}(q_0)$ is an *initial submanifold* of M (cf. [17]).

By Corollary 6.1.14 $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)$ is a smooth embedded submanifold of $\mathcal{O}_{\Delta_R}(q_0)$, hence an initial submanifold of Q. Since $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0) \subset \pi^{-1}_{Q,M}(x_0)$ and $\pi^{-1}_{Q,M}(x_0)$ is diffeomorphic to $\mathrm{Aff}(n)$ using the action μ, we have that $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ is a smooth immersed submanifold of $\mathrm{Aff}(n)$ as well. Now the group multiplication m restricted to $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ which we write as m' is a smooth map $m' : \mathcal{H}^{\nabla}_{\Delta_R|q_0} \times \mathcal{H}^{\nabla}_{\Delta_R|q_0} \to \mathrm{Aff}(n)$ whose image is a subset of $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$. Pulling this map back by the action μ on Q we obtain a smooth map $M : (\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)) \times (\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)) \to Q$ whose image is contained in $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)$. As mentioned above, $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)$ is an initial submanifold of Q, hence M is smooth as a map into $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)$. This then is reflected, by applying the action μ once more, in the fact that m' is smooth as a map into $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$. Thus the latter space is a Lie-subgroup of $\mathrm{Aff}(n)$.

Remark 6.1.16. The situation described in Remark 6.1.5 with the rolling formalism can be treated as the rolling system without spinning nor slipping of two oriented connected Riemannian manifolds (M, g) and $(\mathbb{R}^n, s_n)$, where s_n is the Euclidean metric on $\mathbb{R}^n$. Thus, the state space $Q(M, \mathbb{R}^n)$ is a principal $\mathrm{SE}(n)$-bundle (cf [10], [11] and [12] for more details).

6.2 Case Study: Holonomy of Free Step-two Homogeneous Carnot Group

The goal of this section is to provide an example of a triple (M, ∇, Δ) such that Δ verifies the LARC (and thus is completely controllable) and H^{∇}_{Δ} is a connected compact Lie group strictly included in H^{∇}. After giving the required definitions to treat the example, we first compute H^{∇} and then show the above mentioned propeties of H^{∇}_{Δ} using the rolling formalism.

6.2.1 Definitions

The affine manifold (M, ∇) we consider is the free step-two homogeneous Carnot group $\mathbb{G}_m$ of $m \geq 2$ generators, endowed with a Riemannian metric and its Levi-Civita connection. To describe it, we will use the definitions of Jacobian basis, homogeneous group and Carnot group of Chapters 1 and 2 of [5].

For m positive integer greater than or equal to 2, set $m + n$ where $n := m(m-1)/2$ and $\mathcal{I} := \{(h, k) \mid 1 \leq k < h \leq m\}$ of cardinal n. Let $S^{(h,k)}$ be the $m \times m$ real skew-symmetric matrix whose entries are -1 in the position (h, k), $+1$ in the position (k, h) and 0 elsewhere. On $\mathbb{R}^{m+n}$ where an arbitrary point is written (v, γ) with $v \in \mathbb{R}^m$, and

$\gamma \in \mathbb{R}^n$, define the group law $\star$ by setting

$$(v,\gamma)\star(v',\gamma') = \begin{pmatrix} v_i + v_i', \quad i = 1,\dots,m \\ \gamma_{h,k} + \gamma'_{h,k} + \frac{1}{2}(v_h v_k' - v_k v_h'), \quad (h,k) \in \mathcal{I} \end{pmatrix}. \tag{6.3}$$

Then it is easy to verify that $\mathbb{G}_m := (\mathbb{R}^{m+n},\star)$ is a Lie group, more precisely a free step-two homogeneous Carnot group of m generators. Indeed, a trivial computation shows that the dilation δ_λ given by

$$\delta_\lambda : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}; \quad \delta_\lambda(v,\gamma) = (\lambda v, \lambda^2\gamma), \tag{6.4}$$

is an automorphism of $\mathbb{G}_m$ for every $\lambda > 0$. On the other hand, the (Jacobian) basis of the Lie algebra $\mathfrak{g}_m$ of $\mathbb{G}_m$ is given by X_h, $\Gamma_{h,k}$ where

$$X_h = \frac{\partial}{\partial v_h} + \frac{1}{2}\sum_{1\le j<i\le m}\left(\textstyle\sum_{l=1}^m S_{h,l}^{(i,j)} v_l \right)\left(\frac{\partial}{\partial \gamma_{i,j}}\right),$$

$$= \begin{cases} \frac{\partial}{\partial v_1} + \frac{1}{2}\sum_{1<i\le m} v_i \frac{\partial}{\partial \gamma_{i,1}} & \text{if } h = 1, \\ \frac{\partial}{\partial v_h} + \frac{1}{2}\sum_{h<i\le m} v_i \frac{\partial}{\partial \gamma_{i,h}} - \frac{1}{2}\sum_{1\le j<h} v_j \frac{\partial}{\partial \gamma_{h,j}} & \text{if } 1 < h < m, \\ \frac{\partial}{\partial v_m} - \frac{1}{2}\sum_{1\le j<m} v_j \frac{\partial}{\partial \gamma_{m,j}} & \text{if } h = m, \end{cases}$$

$$\Gamma_{h,k} = \frac{\partial}{\partial \gamma_{h,k}}, \qquad (h,k)\in\mathcal{I}.$$

while the Lie brackets on $\mathbb{G}_m = (\mathbb{R}^{m+n},\star)$ are given by

$$[X_h, X_k] = \sum_{1\le j<i\le m} S_{h,k}^{(i,j)} \frac{\partial}{\partial \gamma_{i,j}} = \frac{\partial}{\partial \gamma_{h,k}} = \Gamma_{h,k}$$
$$[X_h, \Gamma_{i,j}] = 0, \ [\Gamma_{h,k}, \Gamma_{i,j}] = 0.$$

Then,

$$\operatorname{rank}(Lie\{X_1,\dots,X_m\}) = \dim(\operatorname{span}\{\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_m},(\Gamma_{h,k})_{(h,k)\in\mathcal{I}}\}) = m+n = \dim\mathfrak{g}_m.$$

Therefore, we can conclude that $\mathbb{G}_m$ is a homogeneous Carnot group of step 2 and m generators $X_1,\dots,X_m$. The Lie algebra $\mathfrak{g}_m$ is equal to $V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X_1,\dots,X_m\}$ and $V_2 = \operatorname{span}\{\Gamma_{h,k},\ (h,k)\in\mathcal{I}\}$.

Moreover, $(\mathbb{G}_m, g)$ is an analytic manifold where the metric g, with respect to the previous basis, is given by

$$\begin{aligned} &g(X_i,X_j) = \delta_{i,j}, && \text{if } i,j\in\{1,\dots,m\}, \\ &g(X_i,\Gamma_{h,k}) = 0, && \text{if } i\in\{1,\dots,m\} \text{ and } (h,k)\in\mathcal{I}, \\ &g(\Gamma_{h,k},\Gamma_{i,j}) = \delta_{h,i}\delta_{k,j}, && \text{if } (i,j),(h,k)\in\mathcal{I}. \end{aligned} \tag{6.5}$$

In the sequel of this article, we find useful to introduce the following notation of vector fields instead of $\Gamma_{h,k}$, for $h, k \in \{1, \ldots, m\}$, in order to facilitate computations by avoiding the confusion between the two cases $k < h$ and $h < k$.

Definition 6.2.1. *For every $h, k \in \{1, \ldots, m\}$, we define,*

$$\Omega_{h,k} = \begin{cases} \Gamma_{h,k} & \text{if } h > k, \\ -\Gamma_{k,h} & \text{if } h < k, \\ 0 & \text{if } h = k. \end{cases} \tag{6.6}$$

By the above definition, the Lie bracket $[X_h, X_k]$ is equal to $\Omega_{h,k}$, for any $h, k \in \{1, ..., m\}$. Furthermore, let ∇^g be the Levi-Civita connection associated to the Riemannian metric in (6.5).

Lemma 6.2.2. *For $h, k, l, s, t \in \{1, \ldots, m\}$, we have the following covariant derivatives on $(\mathbb{G}_m, g)$,*

$$\nabla^g_{X_h} X_k = \frac{1}{2}\Omega_{h,k}, \qquad \nabla^g_{\Omega_{h,k}} \Omega_{s,t} = 0,$$
$$\nabla^g_{X_l} \Omega_{h,k} = \frac{1}{2}(\delta_{kl} X_h - \delta_{hl} X_k), \qquad \nabla^g_{\Omega_{h,k}} X_l = \frac{1}{2}(\delta_{kl} X_h - \delta_{hl} X_k).$$

Proof. Let us denote by $\nabla^g_X Y$ the covariant differential of a vector field Y in the direction of another vector field X on $\mathbb{G}$. It is equal to

$$\nabla^g_X Y = \sum_{h=1}^{m} \alpha_h(X, Y) X_h + \sum_{1 \le k < h \le m} \beta_{(h,k)}(X, Y) \Omega_{h,k}. \tag{6.7}$$

On the other hand, by Koszul's formula (cf. [32]), we have

$$2g(\nabla^g_X Y, Z) = g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \tag{6.8}$$

Combining (6.7) and (6.8), we easily find the coefficients $\alpha_h(X, Y)$ and $\beta_{(h,k)}(X, Y)$ and hence we obtain the claim.

6.2.2 Riemannian Holonomy Group of $(\mathbb{G}_m, g)$

The main of this subsection is to prove the following theorem.

Theorem 6.2.3. *Let $(\mathbb{G}_m, \nabla^g)$ be a free step-two homogeneous Carnot group of dimension $m + n$ endowed with the Levi-Civita connection ∇^g given in Lemma 6.2.2. Then, $(\mathbb{G}_m, \nabla^g)$ has full holonomy group $H^{\nabla^g} = \mathrm{SO}(m + n)$.*

To this end, we compute the Riemannian tensor curvature R and as well as part of its covariant derivation of $(\mathbb{G}_m, \nabla^g)$.

Lemma 6.2.4. *For any $h,k,l,i,j\in\{1,\dots,m\}$, the Riemannian curvature tensor R of $(\mathbb{G}_m,\nabla^g))$ is given by the following skew-symmetric matrices,*

$$R(X_h,X_k)=\frac{3}{4}(X_h\wedge X_k)+\frac{1}{4}\sum_{j=1}^{m}\Omega_{h,j}\wedge\Omega_{k,j}, \tag{6.9}$$

$$R(X_l,\Omega_{h,k})=\frac{1}{4}(X_h\wedge\Omega_{k,l}+X_k\wedge\Omega_{l,h}), \tag{6.10}$$

$$R(\Omega_{i,j},\Omega_{h,k})=\frac{1}{4}(\delta_{ik}X_h\wedge X_j+\delta_{jk}X_i\wedge X_h+\delta_{ih}X_j\wedge X_k+\delta_{jh}X_k\wedge X_i). \tag{6.11}$$

Proof. From Lemma 6.2.2 and the intrinsic definition of R,

$$R(X,Y)Z=\nabla^g_X\nabla^g_Y Z-\nabla^g_Y\nabla^g_X Z-\nabla^g_{[X,Y]}Z,\qquad \forall X,\ Y,\ Z\in T_x\mathbb{G}_m,$$

we get, for any $h,k,l,i,j\in\{1,\dots,m\}$,

$$R(X_h,X_k)X_l=\tfrac{3}{4}(\delta_{hl}X_k-\delta_{kl}X_h),$$

$$R(X_h,X_k)\Omega_{i,j}=\tfrac{1}{4}(\delta_{ih}\Omega_{k,j}+\delta_{jh}\Omega_{i,k}-\delta_{ik}\Omega_{h,j}-\delta_{jk}\Omega_{i,h}).$$

Similarly, for any $h,k,l,i,j,t\in\{1,\dots,m\}$, $R(X_l,\Omega_{h,k})$ is given by

$$R(X_l,\Omega_{h,k})X_t=\tfrac{1}{4}(\delta_{th}\Omega_{k,l}-\delta_{tk}\Omega_{h,l}),$$

$$R(X_l,\Omega_{h,k})\Omega_{i,j}=\tfrac{1}{4}((\delta_{jk}\delta_{il}-\delta_{jl}\delta_{ki})X_h+(\delta_{jl}\delta_{hi}-\delta_{il}\delta_{jh})X_k).$$

Finally, for any $i,j,h,k,l\in\{1,\dots,m\}$, $R(\Omega_{i,j},\Omega_{h,k})$ is given by

$$\begin{aligned}R(\Omega_{i,j},\Omega_{h,k})X_l=\frac{1}{4}\big(&(\delta_{lk}\delta_{jh}-\delta_{hl}\delta_{jk})X_i+(\delta_{hl}\delta_{ik}-\delta_{lk}\delta_{ih})X_j\\&+(\delta_{il}\delta_{jk}-\delta_{jl}\delta_{ik})X_h+(\delta_{lj}\delta_{ih}-\delta_{hj}\delta_{il})X_k\big),\end{aligned}$$

$$R(\Omega_{i,j},\Omega_{h,k})\Omega_{s,t}=0,\qquad \forall s,t\in\{1,\dots,m\}.$$

Collecting the above equalities, we get Eq. (6.9), Eq. (6.10) and Eq. (6.11).

Using the definition of the covariant derivative of tensors, which is,

$$(\nabla^g_Z R(X,Y))(W)=\nabla^g_Z(R(X,Y)W)-R(X,Y)\nabla^g_Z W,\qquad \forall X,\ Y,\ Z,\ W\in T_x\mathbb{G}_m,$$

we deduce the following lemma.

Lemma 6.2.5. *The covariant derivatives of R in the direction of a vector fields X_t on $\mathbb{G}_m$, for $t\in\{1,\dots,m\}$, are*

$$\nabla^g_{X_t}R(X_h,X_k)=-R(X_t,\Omega_{h,k})+\frac{1}{8}\sum_{j=1}^{m}\big(\delta_{kt}X_j\wedge\Omega_{h,j}-\delta_{ht}X_j\wedge\Omega_{k,j}\big),$$

$$\nabla^g_{X_t}R(X_l,\Omega_{h,k})=\frac{1}{8}\big(\Omega_{t,h}\wedge\Omega_{k,l}+\Omega_{t,k}\wedge\Omega_{l,h}+2\delta_{lt}X_h\wedge X_k+\delta_{ht}X_k\wedge X_l-\delta_{kt}X_h\wedge X_l\big),$$

$$\nabla^g_{X_t}R(\Omega_{i,j},\Omega_{h,k})=\frac{1}{8}\big(\delta_{ik}R(X_j,\Omega_{h,t})+\delta_{jk}R(X_h,\Omega_{i,t})+\delta_{ih}R(X_k,\Omega_{j,t})+\delta_{jh}R(X_i,\Omega_{k,t})\big),$$

where h, k, l, i, j are any integers in $\{1, \dots, m\}$.

Similarly, the covariant derivatives of R in the direction of a vector fields $\Omega_{s,t}$ on $\mathbb{G}_m$, for every $s, t \in \{1, \dots, m\}$, are

$$\begin{aligned}
\nabla^g_{\Omega_{s,t}} R(X_h, X_k) &= \frac{3}{8}\big(\delta_{th} X_s \wedge X_k - \delta_{sh} X_t \wedge X_k + \delta_{tk} X_h \wedge X_s - \delta_{sk} X_h \wedge X_t\big),\\
\nabla^g_{\Omega_{s,t}} R(X_l, \Omega_{h,k}) &= \frac{1}{8}\big(\delta_{th} X_s \wedge \Omega_{k,l} - \delta_{sh} X_t \wedge \Omega_{k,l} + \delta_{tk} X_s \wedge \Omega_{l,h} - \delta_{sk} X_t \wedge \Omega_{l,h}\big),\\
\nabla^g_{\Omega_{s,t}} R(\Omega_{i,j}, \Omega_{h,k}) &= \frac{1}{8}\big((\delta_{ih}\delta_{tk} - \delta_{ik}\delta_{th}) X_j \wedge X_s + (\delta_{ik}\delta_{jt} - \delta_{jk}\delta_{ti}) X_h \wedge X_s\\
&\quad + (\delta_{jh}\delta_{ti} - \delta_{ih}\delta_{tj}) X_k \wedge X_s + (\delta_{jk}\delta_{th} - \delta_{jh}\delta_{tk}) X_i \wedge X_s\\
&\quad - (\delta_{ik}\delta_{js} - \delta_{jk}\delta_{is}) X_h \wedge X_t - (\delta_{jk}\delta_{sh} - \delta_{jh}\delta_{sk}) X_i \wedge X_t\\
&\quad - (\delta_{jh}\delta_{si} - \delta_{ih}\delta_{sj}) X_k \wedge X_t - (\delta_{ih}\delta_{sk} - \delta_{ik}\delta_{sh}) X_j \wedge X_t\big),
\end{aligned}$$

where h, k, l, i, j are any integers in $\{1, \dots, m\}$.

We next deduce from the two previous lemma the main computational result of the section.

Proposition 6.2.6. *Fix some $q_0 \in Q := Q(\mathbb{G}_m, \mathbb{R}^{m+n})$ and let $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$, then $\mathrm{SO}(T_x G_m) \subset \mathcal{O}_{\mathscr{D}_R}(q_0)$.*

Proof. Fix some $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, for any $h, k, i, j \in \{1, \dots, m\}$ such that $i \neq j$ and $k \neq h$, the first order Lie brackets on $\mathcal{O}_{\mathscr{D}_R}(q_0)$ are

$$\begin{aligned}
[\mathscr{L}_R(X_h), \mathscr{L}_R(X_k)]|_q &= \mathscr{L}_R(\Omega_{h,k})|_q + \nu(AR(X_h, X_k))|_q\\
&= \mathscr{L}_R(\Omega_{h,k})|_q + \frac{3}{4}\nu(A(X_h \wedge X_k))|_q + \frac{1}{4}\nu\big(A(\sum_{j=1}^m \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q,\\
[\mathscr{L}_R(\Omega_{i,j}), \mathscr{L}_R(\Omega_{h,k})]|_q &= \frac{1}{4}\nu(A(\delta_{ik} X_h \wedge X_j + \delta_{jk} X_i \wedge X_h + \delta_{ih} X_j \wedge X_k + \delta_{jh} X_k \wedge X_i))|_q,\\
[\mathscr{L}_R(X_i), \mathscr{L}_R(\Omega_{h,k})]|_q &= \frac{1}{4}\nu(A(X_h \wedge \Omega_{k,i} + X_k \wedge \Omega_{i,h}))|_q.
\end{aligned}$$

By taking $i = k$ in the bracket $[\mathscr{L}_R(\Omega_{i,j}), \mathscr{L}_R(\Omega_{h,k})]|_q$, we get that, for any $h, j \in \{1, \dots, m\}$, $\nu(A(X_h \wedge X_j))|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. In addition, from the first and the last brackets of the above Lie brackets, we obtain that $\nu\big(A(\sum_{j=1}^m \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q$ and $\nu(A(X_h \wedge \Omega_{k,i} + X_k \wedge \Omega_{i,h}))|_q$ are tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$, for any $h, k, i, j \in \{1, \dots, m\}$. Thus, we can compute the next bracket in $T_q\mathcal{O}_{\mathscr{D}_R}(q_0)$, for $q \in \mathcal{O}_{\mathscr{D}_R}(q_0)$,

$$\begin{aligned}
[\mathscr{L}_R(X_i), \nu((\cdot)(X_h \wedge X_k))]|_q &= \delta_{ki}\mathscr{L}_{NS}(AX_h)|_q - \delta_{hi}\mathscr{L}_{NS}(AX_k)|_q\\
&- \tfrac{1}{2}\nu(A(X_h \wedge \Omega_{k,i} + X_k \wedge \Omega_{i,h}))|_q.
\end{aligned}$$

Using $[\mathscr{L}_R(X_i), \mathscr{L}_R(\Omega_{h,k})]|_q$ and then putting $i = h$ in the last Lie bracket, we obtain that $\mathscr{L}_{NS}(AX_k)|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$, for all $k \in \{1, \dots, m\}$. In addition, we have

$$\begin{aligned}
& [\mathscr{L}_R(\Omega_{t,s}), \nu((\cdot)(X_h \wedge \Omega_{k,i} + X_k \wedge \Omega_{i,h}))]|_q \\
= \; & (\delta_{tk}\delta_{ls} - \delta_{tl}\delta_{sk})\mathscr{L}_{NS}(AX_h)|_q + (\delta_{ht}\delta_{ls} - \delta_{tl}\delta_{hs})\mathscr{L}_{NS}(AX_k)|_q \\
+ \; & \tfrac{1}{2}\big(\delta_{sh}\nu(A(X_t \wedge \Omega_{k,l}))|_q + \delta_{sk}\nu(A(X_t \wedge \Omega_{l,h}))|_q \\
& \quad -\delta_{th}\nu(A(X_s \wedge \Omega_{k,l}))|_q + \delta_{tk}\nu(A(X_s \wedge \Omega_{l,h}))|_q\big).
\end{aligned}$$

Since $\mathscr{L}_{NS}(AX_k)|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$, for all $k \in \{1, \dots, m\}$, then $\nu(A(X_t \wedge \Omega_{h,k}))|_q$ is also tangent for any distinct integers $h, k, t \in \{1, \dots, m\}$. The last Lie bracket to compute is

$$\begin{aligned}
[\mathscr{L}_{NS}(X_t), \nu\big(A(X_l \wedge \Omega_{h,k})\big)]|_q = \frac{1}{2}\big(&\delta_{tk}\nu(A(X_l \wedge X_h))|_q - \delta_{th}\nu(A(X_l \wedge X_k))|_q \\
&+ \nu(A(\Omega_{t,l} \wedge \Omega_{h,k}))|_q\big).
\end{aligned}$$

Therefore, for every $h, k, t, l \in \{1, \dots, m\}$, $\nu(A(\Omega_{t,l} \wedge \Omega_{h,k}))|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Hence, for all $q \in \mathcal{O}_{\mathscr{D}_R}(q_0)$ the following vector fields

$$\nu(A(X_h \wedge X_k))|_q, \quad \nu(A(X_t \wedge \Omega_{h,k}))|_q, \quad \nu(A(\Omega_{t,l} \wedge \Omega_{h,k}))|_q,$$

are tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. This completes the proof because we have that $\nu(AB)|_q \in T_q\mathcal{O}_{\mathscr{D}_R}(q_0)$ if and only if $B \in \mathfrak{so}(T_x\mathbb{G}_m)$ for $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathscr{D}_R}(q_0)$.

We return to prove the main theorem in the beginning of the current subsection.

Proof. [Proof of Theorem 6.2.3]. As the vertical bundle of Q is included in the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ by Proposition 6.2.6, then the rolling problem $(\Sigma)_R$ is completely controllable (see Corollary 5.21 in [10]). According to Theorem 4.3 in [12], the holonomy group of $\mathbb{G}_m$ is equal to $\mathrm{SO}(m+n)$. Note that one could have used as well the main result in [29] stating that the tangent space of the holonomy group at every point $x \in M$ contains the evaluations at x of the curvature tensor and its covariant derivatives at any order.

6.2.3 Horizontal Holonomy Group of $(\mathbb{G}_m, g)$

We define the distribution $\Delta := \mathrm{span}\{X_1, \dots, X_m\}$ on $\mathbb{G}_m$ and $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. Note that is of constant rank m. We will first compute a basis of $T_q\mathcal{O}_{\Delta_R}(q_0)$ for any $q \in \mathcal{O}_{\Delta_R}(q_0)$ and then investigate the holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ of rolling of $(\mathbb{G}_m, g)$ against $(\mathbb{R}^{m+n}, s_{m+n})$, where s_{m+n} is the Euclidean metric on $\mathbb{R}^{m+n}$.

The Tangent Space of $\mathcal{O}_{\Delta_R}(q_0)$

Proposition 6.2.7. *For any $q_0 \in Q$, the tangent space of $\mathcal{O}_{\Delta_R}(q_0)$ is generated by the following linearly independent vector fields:*

$$\mathscr{L}_{NS}(X_h)|_q,\ \mathscr{L}_{NS}(AX_h)|_q,\ \mathscr{L}_{NS}(A\Omega_{h,k})|_q,\ \mathscr{L}_{NS}(\Omega_{h,k})|_q + \frac{1}{2}\nu(A(X_h \wedge X_k))|_q, \tag{6.12}$$
$$\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q,\ \nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q.$$

Proof. Recall that all the data of the problem are analytic. Then, by the analytic version of the orbit theorem of Nagano-Sussmann (cf. [2]), the orbit $\mathcal{O}_{\Delta_R}(q_0)$ is an immersed analytic submanifold in the state space Q and $T_q\mathcal{O}_{\Delta_R}(q_0) = Lie_q(\Delta_R)$. Therefore, we are left to determine vector fields spanning $Lie_q(\Delta_R)$, i.e., to compute enough iterated Lie brackets of Δ.

For any $h, k, l, s, t, p \in \{1, \ldots, m\}$, we have

$$\begin{aligned} & [\mathscr{L}_R(X_h), \mathscr{L}_R(X_k)]|_q = \mathscr{L}_R(\Omega_{h,k})|_q + \nu(AR(X_h, X_k))|_q \\ = \;& \mathscr{L}_R(\Omega_{h,k})|_q + \tfrac{3}{4}\nu(A(X_h \wedge X_k))|_q + \tfrac{1}{4}\nu\big(A(\textstyle\sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q. \end{aligned} \tag{6.13}$$

$$\begin{aligned} & [\mathscr{L}_R(X_l), [\mathscr{L}_R(X_h), \mathscr{L}_R(X_k)]]|_q \\ = \;& \delta_{kl}\ \big(\tfrac{3}{4}\mathscr{L}_{NS}(AX_h)|_q + \tfrac{1}{8}\nu(A\textstyle\sum_{j=1}^{m} X_j \wedge \Omega_{h,j}))|_q\big) \\ - \;& \delta_{hl}\ \big(\tfrac{3}{4}\mathscr{L}_{NS}(AX_k)|_q + \tfrac{1}{8}\nu(A(\textstyle\sum_{j=1}^{m} X_j \wedge \Omega_{k,j}))|_q\big). \end{aligned} \tag{6.14}$$

$$\begin{aligned} & [\mathscr{L}_R(X_t), [\mathscr{L}_R(X_l), [\mathscr{L}_R(X_h), \mathscr{L}_R(X_k)]]]|_q \\ = \;& \delta_{kl}\ \big(\tfrac{1}{2}\mathscr{L}_{NS}(A\Omega_{t,h})|_q + \tfrac{1}{16}\nu(A(X_t \wedge X_h))|_q + \tfrac{1}{16}\nu(A(\textstyle\sum_{j=1}^{m} \Omega_{t,j} \wedge \Omega_{h,j}))|_q\big) \\ - \;& \delta_{hl}\ \big(\tfrac{1}{2}\mathscr{L}_{NS}(A\Omega_{t,k})|_q + \tfrac{1}{16}\nu(A(X_t \wedge X_k))|_q + \tfrac{1}{16}\nu(A(\textstyle\sum_{j=1}^{m} \Omega_{t,j} \wedge \Omega_{k,j}))|_q\big). \end{aligned} \tag{6.15}$$

$$\begin{aligned}
&[\mathscr{L}_R(X_s),[\mathscr{L}_R(X_t),[\mathscr{L}_R(X_l),[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]]]]|_q \\
= \;& \delta_{kl}\Big(\delta_{hs}\big(\tfrac{3}{8}\mathscr{L}_{NS}(AX_t)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{t,j}))\big) \\
&\quad -\delta_{ts}\big(\tfrac{3}{8}\mathscr{L}_{NS}(AX_h)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{h,j}))\big)\Big) \\
- \;& \delta_{hl}\Big(\delta_{ks}\big(\tfrac{3}{8}\mathscr{L}_{NS}(AX_t)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{t,j}))\big) \\
&\quad -\delta_{ts}\big(\tfrac{3}{8}\mathscr{L}_{NS}(AX_k)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{k,j}))\big)\Big).
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
&[\mathscr{L}_R(X_p),[\mathscr{L}_R(X_s),[\mathscr{L}_R(X_t),[\mathscr{L}_R(X_l),[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]]]]]|_q \\
= \;& \delta_{kl}\Big(\delta_{hs}\big(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,t})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_t))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{t,j}))|_q\big) \\
&\quad -\delta_{ts}\big(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,h})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_h))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{h,j}))|_q\big)\Big) \\
- \;& \delta_{hl}\Big(\delta_{ks}\big(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,t})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_t))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{t,j}))|_q\big) \\
&\quad -\delta_{ts}\big(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,k})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_k))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{k,j}))|_q\big)\Big).
\end{aligned} \tag{6.17}$$

First we should remark that, by iteration, the commutators

$$[\mathscr{L}_R(X_{\alpha_r}),\dots[\mathscr{L}_R(X_{\alpha_2}),\mathscr{L}_R(X_{\alpha_1})]\dots],$$

where $\alpha_i\in\{1,\dots,m\}$ and $r\geq 3$ are written either as the vectors in (6.14) and (6.16), or as those in (6.15) and (6.17). Therefore, one only has to prove that (6.13), (6.14), (6.15), (6.16) and (6.17) are linear combinations of the vector fields in (6.12) and then show that the Lie algebra generated by these vector fields is involutive. Indeed, fix some $h,k\in\{1,\dots,m\}$ such that $k\neq h$ and take $p=s=t=l=k$ in the above Lie brackets. Calculate $\frac{1}{4}(6.14)+(6.16)$ and $\frac{1}{2}(6.14)+(6.16)$, we get that

$$\mathscr{L}_{NS}(AX_h)|_q,\quad \nu(A(\sum_{j=1}^m X_j\wedge\Omega_{h,j}))|_q$$

are vectors in $Lie(\Delta_R)$. On the other hand, $\frac{1}{4}(6.15)+(6.17)$ and $\frac{7}{16}(6.15)+(6.17)$ imply that

$$\mathscr{L}_{NS}(A\Omega_{h,k})|_q,\quad \nu(A(X_h\wedge X_k+\sum_{j=1}^m \Omega_{h,j}\wedge\Omega_{k,j}))|_q$$

belong also to $Lie(\Delta_R)$. The last two vectors with (6.13) give us another vector in $Lie(\Delta_R)$ which is

$$\mathscr{L}_{NS}(\Omega_{h,k})|_q + \frac{1}{2}\nu(A(X_h \wedge X_k))|_q.$$

To show that the vector fields given in Eq. (6.12) form a basis for $Lie(\Delta_R)$, it remains to compute their first order Lie brackets to see that they define an involutive Lie algebra.

We have,

$$\begin{aligned} & [\nu((\cdot)(\textstyle\sum_{j=1}^m X_j \wedge \Omega_{h,j})), \nu((\cdot)(\textstyle\sum_{j=1}^m X_j \wedge \Omega_{k,j}))]|_q \\ = \;& \nu(A(X_h \wedge X_k + \textstyle\sum_{j=1}^m \Omega_{h,j} \wedge \Omega_{k,j}))|_q, \end{aligned} \tag{6.18}$$

$$\begin{aligned} & [\nu((\cdot)(\textstyle\sum_{j=1}^m X_j \wedge \Omega_{l,j})), \nu((\cdot)(X_h \wedge X_k + \textstyle\sum_{j=1}^m \Omega_{h,j} \wedge \Omega_{k,j}))]|_q \\ = \;& \delta_{kl}\nu(A(\textstyle\sum_{j=1}^m X_j \wedge \Omega_{h,j}))|_q - \delta_{hl}\nu(A(\textstyle\sum_{j=1}^m X_j \wedge \Omega_{k,j}))|_q, \end{aligned} \tag{6.19}$$

and,

$$\begin{aligned} & [\nu((\cdot)(X_l \wedge X_t + \textstyle\sum_{j=1}^m \Omega_{l,j} \wedge \Omega_{t,j})), \nu((\cdot)(X_h \wedge X_k + \textstyle\sum_{j=1}^m \Omega_{h,j} \wedge \Omega_{k,j}))]|_q \\ = \;& \delta_{kl}\nu(A(X_h \wedge X_t + \textstyle\sum_{j=1}^m \Omega_{h,j} \wedge \Omega_{t,j}))|_q + \delta_{hl}\nu(A(X_t \wedge X_k + \textstyle\sum_{j=1}^m \Omega_{t,j} \wedge \Omega_{k,j}))|_q \\ + \;& \delta_{tk}\nu(A(X_l \wedge X_h + \textstyle\sum_{j=1}^m \Omega_{l,j} \wedge \Omega_{h,j}))|_q + \delta_{ht}\nu(A(X_k \wedge X_l + \textstyle\sum_{j=1}^m \Omega_{k,j} \wedge \Omega_{l,j}))|_q. \end{aligned} \tag{6.20}$$

Moreover, the Lie brackets between $\mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu(A(X_h \wedge X_k))$ and the remaining vectors of (6.12) are

$$\begin{aligned} & [\mathscr{L}_{NS}(X_l), \mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu((\cdot)(X_h \wedge X_k))]|_q = 0, \\ & [\mathscr{L}_{NS}((\cdot)X_l), \mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu((\cdot)(X_h \wedge X_k))]|_q = 0, \\ & [\mathscr{L}_{R}((\cdot)\Omega_{i,j}), \mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu((\cdot)(X_h \wedge X_k))]|_q = 0, \\ & [\mathscr{L}_{NS}((\cdot)\Omega_{l,t}), \mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu((\cdot)(X_h \wedge X_k))]|_q = 0, \\ & [\mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu((\cdot)(X_h \wedge X_k)), \mathscr{L}_{NS}(\Omega_{l,t}) + \frac{1}{2}\nu((\cdot)(X_l \wedge X_t))]|_q = 0, \\ & [\mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu((\cdot)(X_h \wedge X_k)), \nu((\cdot)(X_l \wedge X_t + \sum_{j=1}^m \Omega_{l,j} \wedge \Omega_{t,j}))]|_q = 0, \\ & [\mathscr{L}_{NS}(\Omega_{h,k}) + \frac{1}{2}\nu((\cdot)(X_h \wedge X_k)), \nu((\cdot)(\sum_{j=1}^m X_j \wedge \Omega_{l,j}))]|_q = 0. \end{aligned}$$

Then, the vector fields

$$\mathscr{L}_{NS}(X_h)|_q,\ \mathscr{L}_{NS}(AX_h)|_q,\ \mathscr{L}_{NS}(A\Omega_{h,k})|_q,\ \mathscr{L}_{NS}(\Omega_{h,k})|_q + \frac{1}{2}\nu(A(X_h \wedge X_k))|_q,$$
$$\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q,\ \nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q,$$

form an involutive distribution and any vector fields in $Lie(\Delta_R)$ is a linear combination of them. It remains to check that they are linearly independent. It is clearly enough to do that for the family of vector fields $\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q$. Suppose there exists $(\alpha_h)_{1\leq h\leq m}$, such that $\sum_{h=1}^{m} \alpha_h \sum_{j=1}^{m} X_j \wedge \Omega_{h,j} = 0$. Then $\sum_{j=1}^{m} X_j \wedge (\sum_{h=1,h\neq j}^{m} \alpha_h \Omega_{h,j}) = 0\ \forall j \in \{1,\dots,m\}$. Hence, $\sum_{h=1,h\neq j}^{m} \alpha_h \Omega_{h,j} = 0$ for every $j,h \in \{1,\dots,m\}$, so $\alpha_h = 0$ for every $h \in \{1,\dots,m\}$. Therefore,

$$\mathscr{L}_{NS}(X_h)|_q,\ \mathscr{L}_{NS}(AX_h)|_q,\ \mathscr{L}_{NS}(A\Omega_{h,k})|_q,\ \mathscr{L}_{NS}(\Omega_{h,k})|_q + \frac{1}{2}\nu(A(X_h \wedge X_k))|_q,$$
$$\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q,\ \nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q,$$

is a global basis of $Lie_q(\Delta_R)$ and hence the dimension of $Lie_q(\Delta_R)$ is constant and equal to $3(m+n)$. We deduce that $\dim \mathcal{O}_{\Delta_R}(q_0) = 3(m+n)$ and the tangent space of $\mathcal{O}_{\Delta_R}(q_0)$ is generated by the vectors in (6.12).

Remark 6.2.8. According to this proposition, $\mathscr{L}_{NS}(AX_h)|_q$ and $\mathscr{L}_{NS}(A\Omega_{h,k})|_q$ are tangent to $\mathcal{O}_{\Delta_R}(q_0)$. This implies that $\pi_{Q,\mathbb{R}^N}(\mathcal{O}_{\Delta_R}) = \mathbb{R}^{m+n}$ which means that all the translations along $\mathbb{R}^{m+n}$ are included in the tangent space of the orbit. Furthermore, the families of vector fields $\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q$ and $\nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q$ form an involutive vertical distribution.

Study of $\mathcal{H}^{\nabla}_{\Delta_R|q}$

The main result of the subsection is given next.

Proposition 6.2.9. *(i) The affine Δ-horizontal holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ is a connected Lie subgroup of* $\mathrm{SE}(m+n)$ *of dimension* $2(m+n)$.

(ii) The Δ-horizontal holonomy group H^{∇}_{Δ} is a connected compact Lie subgroup of $\mathrm{SO}(m+n)$ *of dimension* $m+n$.

In particular, the inclusions $clos(\mathcal{H}^{\nabla}_{\Delta_R}) \subset clos(\mathcal{H}^{\nabla})$ and $clos(H^{\nabla}_{\Delta}) \subset clos(H^{\nabla})$ are strict if and only if $m \geq 3$.

Proof. As an immediate adaptation of Proposition 6.1.15 to the case where one is dealing with principal $\mathrm{SE}(m+n)$-bundles, one gets that the affine holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ is a Lie subgroup of $\mathrm{SE}(m+n)$. Notice then that, if $\Pi : \mathrm{SE}(m+n) \to \mathrm{SO}(m+n)$ is the projection onto the $\mathrm{SO}(m+n)$ factor of $\mathrm{SE}(m+n)$, one has, by definition $H^{\nabla} = \Pi(\mathcal{H}^{\nabla})$ and $H^{\nabla}_{\Delta} = \Pi(\mathcal{H}^{\nabla}_{\Delta_R})$. This shows that the Δ-horizontal holonomy group H^{∇}_{Δ} is a Lie subgroup of $\mathrm{SO}(m+n)$.

We next prove that for every $q' \in Q$, a basis of $\mathrm{Lie}(\mathcal{H}^{\nabla}_{\Delta_R|q'})$ the Lie-algebra of $\mathcal{H}^{\nabla}_{\Delta_R|q'}$ is given by the evaluation at q' of the vector fields whose values at $q = (x, \hat{x}, A) \in Q$ are

$$\mathscr{L}_{NS}(AX_h)|_q, \mathscr{L}_{NS}(A\Omega_{h,k})|_q, \nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q, \nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q, \tag{6.21}$$

and hence this Lie algebra has dimension $2(m+n) = m(m+1)$. To see that, consider some element $V \in \mathrm{Lie}(\mathcal{H}^{\nabla}_{\Delta_R|q'})$ as a linear subspace of $T_{q'}\mathcal{O}_{\Delta_R}(q_0)$. Then V is a linear combination of the vector fields described in Eq. (6.12) evaluated at q' and V projects to a zero-vector in TM. By an obvious computation, one deduces that V is a linear combination of the vector fields given in Eq. 6.21. Conversely, it is clear that the vector fields given in Eq. (6.21) generate a distribution whose integral manifolds lie in $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x')$ where $x' = \pi_{Q,M}(q')$. This proves that $\mathrm{Lie}(\mathcal{H}^{\nabla}_{\Delta_R|q'})$ the Lie-algebra of $\mathcal{H}^{\nabla}_{\Delta_R|q'}$ has dimension $2(m+n) = m(m+1)$. One could also check that the distribution generated by the vector fields in Eq. (6.21) is involutive.

By a similar reasoning, a basis $\mathrm{Lie}(H^{\nabla}_{\Delta|q'})$ of the Lie-algebra of $H^{\nabla}_{\Delta|q'}$ is given by the (evaluations at q' of) vector fields (see also (6.22) below)

$$\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q,\ \nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q,$$

and hence this Lie algebra has dimension $m + n = m(m+1)/2$.

It is proved in Subsection 6.3.2 that $\mathcal{H}^{\nabla}_{\Delta_R}$ is connected and, as a consequence, H^{∇}_{Δ} is connected as well.

It remains to prove the rest of Claim (ii). For $1 \leq h \leq m$, let $A_h \in \mathfrak{so}(m+n)$ corresponding to the vertical vector $\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)$ and, for $(h,k) \in \mathcal{I}$, let $B_{h,k} \in \mathfrak{so}(m+n)$ corresponding to the vertical vector $\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)$. We extend the notations for the $B_{h,k}$ to for any $1 \leq h,k \leq m$ by setting $B_{h,k} = -B_{k,h}$. The basis of Lie algebra $L := \mathrm{Lie}(H^{\nabla}_{\Delta})$ of H^{∇}_{Δ} is given by the matrices A_h, $1 \leq h \leq m$ and $B_{h,k}$, $(h,k) \in \mathcal{I}$ and thanks to Eqs. (6.18), (6.19) and (6.20), one has

$$[A_i, A_j] = B_{i,j},\ [A_i, B_{h,k}] = \delta_{ki}A_h - \delta_{hi}A_k,\ [B_{l,t}, B_{h,k}] = \delta_{kl}B_{h,t} + \delta_{hl}B_{t,k} + \delta_{tk}B_{l,h} + \delta_{ht}B_{k,l}. \tag{6.22}$$

We next prove that $L = \mathfrak{so}(m+1)$ and, for that purpose, we build a Lie algebra isomorphism φ between L and $\mathfrak{so}(m+1)$. Let $(e_j)_{1 \leq j \leq m+1}$ be an orthonormal basis of $\mathbb{R}^{m+1}$. To each $(h,k) \in \mathcal{I}$ and $1 \leq t \leq m$, define $\varphi(B_{h,k}) = e_k \wedge e_h$ and $\varphi(A_t) = e_t \wedge e_{m+1}$ and then extend the definition of φ by linearity to the whole L. It is then immediate to check that φ is a Lie algebra isomorphism.

Recall that H^{∇}_{Δ} is connected. Since its Lie algebra L is semi-simple and compact, it follows from Weyl's theorem (cf. [30, Theorem 26.1]) that H^{∇}_{Δ} is compact in $\mathrm{SO}(m+n)$ and of dimension $m + n$.

6.3 Appendix

6.3.1 *o-regular* controls

We first generalize the usual definition of regular control and then provide a result about existence of such controls. Let M be an n-dimensional smooth manifold, $\mathcal{F}$ a (possibly infinite) family of smooth vector fields on M, and let $\Delta_{\mathcal{F}}$ be the smooth *singular* distribution (cf. [17]) spanned by $\mathcal{F}$, i.e.

$$\Delta_{\mathcal{F}}|_p = \operatorname{span}\{X|_p \mid X \in \mathcal{F}\} \subset T_pM, \quad p \in M.$$

We use the word "singular" (to emphasize the fact that the rank (dimension) of $\Delta_{\mathcal{F}}$ might vary from point to point. One can, in fact, prove given any such family $\mathcal{F}$, there is a *finite* subfamily $\mathcal{F}_0 = \{X_1, \ldots, X_m\}$ such that $\Delta_{\mathcal{F}} = \Delta_{\mathcal{F}_0}$, and $m \leq n(n+1)$ (see [13, 35], or [31] when $\Delta_{\mathcal{F}}$ has constant rank). Moreover, by span S we mean $\mathbb{R}$-linear span of a set S.

Definition 6.3.1. *An absolutely continuous (a.c.) curve $\gamma : [0,T] \to M$ is horizontal with respect to $\mathcal{F}$ if there is a finite subfamily $\{X_1, \ldots, X_m\}$ of $\mathcal{F}$ and $u = (u_1, \ldots, u_d) \in L^1([0,T], \mathbb{R}^m)$, $m \in \mathbb{N}$ (here m might depend on the curve γ in question), such that for almost every $t \in [0,T]$,*

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i(t) X_i|_{\gamma(t)}.$$

The orbit *$\mathcal{O}_{\mathcal{F}}(p)$ of $\mathcal{F}$ through $p \in M$ is the set of all points of M reached by $\mathcal{F}$-horizontal paths γ with $\gamma(0) = p$.*

If Δ is a smooth distribution of *constant rank* k on M, and if $\mathcal{F} = \mathcal{F}_\Delta$ is the set of smooth vector fields tangent to Δ, then it is easy to see that $\Delta = \Delta_{\mathcal{F}}$, and that an a.c. curve is Δ-horizontal if and only if it is $\mathcal{F}$-horizontal. Therefore, in this case the concept of orbit coincides with the notion we have used previously in the paper, and one can without ambiguity denote it by $\mathcal{O}_\Delta(p)$ instead of $\mathcal{O}_{\mathcal{F}}(p)$.

For a smooth vector field X write $\Phi_X : D \to M$ for its flow, where $D = D_X$ is an open connected subset of $\mathbb{R} \times M$ containing $\{0\} \times M$. We also use the notation $(\Phi_X)_t(x) = (\Phi_X)^x(t) = \Phi_X(t,x)$ when $(x,t) \in D$.

The orbit of a family $\mathcal{F}$ of vector fields has the following properties (cf. [17], [21]).

Theorem 6.3.2 (Orbit Theorem).

1. *The orbit $\mathcal{O}_{\mathcal{F}}(p)$ is an immersed submanifold of M.*
2. *Any continuous (resp. smooth) map $f : Z \to M$, where Z is a smooth manifold, such that $f(Z) \subset \mathcal{O}_{\mathcal{F}}(p)$ is continuous (resp. smooth) as a map $f : Z \to \mathcal{O}_{\mathcal{F}}(p)$.*
3. *If one writes $G_{\mathcal{F}}$ for the set of all locally defined diffeomorphisms of M of the form $(\Phi_{X_r})^{t_1} \circ \cdots \circ (\Phi_{X_d})^{t_d}$ for $X_1, \ldots, X_d \in \mathcal{F}$ and $t_1, \ldots, t_d \in \mathbb{R}$ for which this*

map is defined, then

$$\mathcal{O}_{\mathcal{F}}(p) = \{\varphi(p) \mid \varphi \in G_{\mathcal{F}}\}$$
$$T\mathcal{O}_{\mathcal{F}}(p) = \operatorname{span}\{\varphi_*(X) \mid \varphi \in G_{\mathcal{F}},\ X \in \mathcal{F}\},$$

wherever the expressions $\varphi(p)$ and $\varphi_*(X)$ are defined.

As a consequence of Item 3. of the theorem, one sees that $L^1([0,T],\mathbb{R}^m)$ in Definition 6.3.1 can be replaced by $L^2([0,T],\mathbb{R}^m)$, which for the rest paper will be the appropriate *space of controls* for our needs.

Following [31] we define the concepts of the end-point mapping and that of a regular (L^2-)control.

Definition 6.3.3. *For every $p \in M$, any time $T > 0$, and any smooth finite family of vector fields $\mathcal{F} = \{X_1, \dots, X_m\}$ on M, there exists a maximal open subset $U_{\mathcal{F}}^{p,T} \subset L^2([0,T],\mathbb{R}^m)$ such that for every $u = (u_1, \dots, u_m) \in U_p^T$, there exists a unique absolutely continuous solution $\gamma_u : [0,T] \to M$ to the Cauchy problem*

$$\dot{\gamma_u}(t) = \sum_{i=1}^{m} u_i(t) X_i(\gamma_u(t)), \quad \gamma_u(0) = p. \tag{6.23}$$

The end-point map $E_{\mathcal{F}}^{p,T}$ associated to $\mathcal{F}$ at p in time T is defined as the mapping

$$E_{\mathcal{F}}^{p,T} : U_{\mathcal{F}}^{p,T} \to M, \quad E_{\mathcal{F}}^{p,T}(u) = \gamma_u(T).$$

By [31, Proposition 1.8] we have the following.

Proposition 6.3.4. *With $p, T, \mathcal{F}$ as above, the end point map $E_{\mathcal{F}}^{p,T} : U_{\mathcal{F}}^{p,T} \to M$ is C^1-smooth.*

This proposition allows us to give the following definition.

Definition 6.3.5. *A control $u \in U_{\mathcal{F}}^{p,T}$ is said to be* o-regular *with respect to p in time T if the rank of $D_u E_{\mathcal{F}}^{p,T} : L^2([0,T],\mathbb{R}^m) \to T_{E_{\mathcal{F}}^{p,T}(u)} M$, the differential of $E_{\mathcal{F}}^{p,T}(u)$ at u, is equal to* $\dim \mathcal{O}_{\mathcal{F}}(p)$. *Here, "o-regular" stands for orbitally regular.*

Remark 6.3.6. A control u is usually said to be regular (with respect to p in time T) if the rank of $E_{\mathcal{F}}^{p,T}(u)$ is equal to the dimension n of the ambient manifold M (cf [31, Section 1.3]), implying in particular that the orbit $\mathcal{O}_{\mathcal{F}}(p)$ is open in M and thus is n-dimensional. If the distribution generated by $\mathcal{F}$ verifies the LARC, it can be proved that any pair of points in M can be joined by the trajectory tangent to this distribution and corresponding to a regular control, cf. [3]. In this paper, we have extended this definition without assuming controllability.

The main purpose of this appendix is to generalize the result of [3] to the case where the distribution Δ is not necessarily bracket-generating. Indeed, we have the following result.

Proposition 6.3.7. *Let M be an n-dimensional smooth manifold, $\mathcal{F} = \{X_1, \dots, X_m\}$, $m \in \mathbb{N}$, a smooth* finite *family of vector fields on M. Then, for every $p \in M$ and time $T > 0$, and every $q \in \mathcal{O}_{\mathcal{F}}(p)$, there exists an* o-regular *control with respect to p in time T such that the unique solution γ_u to the Cauchy problem* (6.23) *such that $\gamma_u(T) = q$.*

Remark 6.3.8. By the proof of Proposition 1.12 in [31] (see also [17]), the conclusion is immediate if $T_q\mathcal{O}_{\mathcal{F}}(p)$ is equal for every $q \in \mathcal{O}_{\mathcal{F}}(p)$ to $\mathrm{Lie}_q(\mathcal{F})$, the evaluation at q of the Lie algebra generated by $\mathcal{F}$. In fact, in this case a stronger result holds, namely the set of regular controls is dense in $U_{\mathcal{F}}^{q,T}$ for every $q \in \mathcal{O}_{\mathcal{F}}(p)$ and $T > 0$. As a consequence, any control $u_0 \in E_{\mathcal{F}}^{p,T}$ admits an o-regular control u arbitrarily close (in L^2) to u_0 such that $E_{\mathcal{F}}^{p,T}(u) = E_{\mathcal{F}}^{p,T}(u_0)$.

Proof. Fix $q_0 \in \mathcal{O}_{\mathcal{F}}(p)$ and $(Z_1^0, \dots, Z_d^0)$ a basis of $T_{q_0}\mathcal{O}_{\mathcal{F}}(p)$. According to Theorem 6.3.2, there exists $\varphi_1 \in G_{\mathcal{F}}$ and $Y_1 \in \mathcal{F}$ with $q_1 := \varphi_1^{-1}(q_0)$ such that $Z^0 := (\tilde{Z}_1^0, Z_2^0, \dots, Z_d^0)$, where $\tilde{Z}_1^0 = (\varphi_1)_* Y_1|_{q_0}$, forms a basis of $\mathcal{O}_{\mathcal{F}}(p)$ at q_0. The basis Z^0 is the pushforward of a basis $Z^1 = (Z_1^1, \dots, Z_d^1)$ of $T_{q_1}\mathcal{O}_{\mathcal{F}}(p)$ by φ_1 and obviously $Z_1^1 = Y_1$. We proceed inductively (using Theorem 6.3.2) with this construction for $1 \leq l \leq d$ so that the basis $Z^{l-1} = (Z_1^{l-1}, \dots, Z_{l-1}^{l-1}, \tilde{Z}_l^{l-1}, Z_{l+1}^{l-1}, \dots, Z_d^{l-1})$ of $T_{q_{l-1}}\mathcal{O}_{\mathcal{F}}(p)$ is the pushforward of a basis $Z^l = (Z_1^l, \dots, Z_d^l)$ with $q_l := \varphi_l^{-1}(q_{l-1})$, $Y_l := Z_l^l \in \mathcal{F}$ and $\tilde{Z}_l^{l-1} = (\varphi_l)_*(Z_l^l)$. Finally consider $\varphi_{d+1} \in G$ so that $\varphi_{d+1}(p) = q_n$ and set $\psi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{d+1}$. One has that $\psi(p) = q_0$ and there exists $T > 0$ and $u \in L^2([0,T], \mathbb{R}^m)$ such that the unique solution γ_u to the Cauchy problem $\dot{\gamma}_u(t) = \sum_{i=1}^m u_i(t) X_i|_{\gamma_u(t)}$, $x(0) = p$ verifies $\gamma_u(T) = q_0$. Then the flow of diffeomorphisms $\psi^u(t,q)$ corresponding to the time-varying vector field $q \mapsto \sum_{i=1}^m u_i(t) X_i|_q$ verifies $\psi = \psi^u(T,0)$ and $\psi^u(t,p) = \gamma_u(t)$ where one has, for $0 \leq s \leq t \leq T$, $\frac{\partial \psi^u(t,q)}{\partial t} = \sum_{i=1}^m u_i(t) X_i|_{\psi^u(t,q)}$ together with the initial condition $\psi^u(0,q) = q$ for every $q \in M$. With the above notations, it is clear that, for every $1 \leq l \leq d$,

$$\left(d_q\psi^u(T,p)(d_q\psi^u(t_l,p))^{-1}Y_l\right)_{l=1,\dots,d} =: (\tilde{Z}_1, \dots, \tilde{Z}_d)$$

forms a basis of $T_{q_0}\mathcal{O}_{\mathcal{F}}(p)$, where $d_q\psi^u(t,\cdot)$ denotes the differential of $\psi(t,q)$ with respect to the q variable.

Recall that the differential of the end-point map at u is the linear map $D_u E_{\mathcal{F}}^{p,T} : L^2([0,T], \mathbb{R}^m) \to T_{q_0}\mathcal{O}_{\mathcal{F}}(p)$ given by

$$D_u E_{\mathcal{F}}^{p,T}(v) = d_q\psi^u(T,p) \int_0^T (d_q\psi^u(t,p))^{-1} X^v(t, \gamma_u(t)) dt, \tag{6.24}$$

where $X^v(t,x) = \sum_{i=1}^m v_i(t) X_i|_x$ for almost every $t \in [0,T]$ and every $x \in M$. We further complete the notations as follows. Let $0 = t_0 < t_1 < \cdots < t_{d+1} := T$ the sequence of times where $\gamma_u(t_l) = q_{d+1-l}$ with the convention that $p = q_{d+1}$ and thus $\psi^u(T,p)\psi^u(t_l,p)^{-1}(q_l) = q_0$, for $0 \leq l \leq d+1$. Moreover, one has $Y_l = \sum_{i=1}^m y_{il} X_i|_{q_l}$ for $1 \leq l \leq d$ and some real numbers (y_{il}).

For every $\varepsilon > 0$ small enough and $1 \leq l \leq d$, consider the sequence (v_ε^l) of functions in $L^2([0,T], \mathbb{R}^m)$ defined by $v_\varepsilon^l(t) = \frac{1}{\varepsilon}(y_{il})_{1 \leq i \leq k}$ if $t_l - \varepsilon \leq t \leq t_l$ and zero otherwise. It is a matter of standard computations (as performed in [31, Proposition 1.10]) to prove that, for every $1 \leq l \leq d$, $D_u E_{\mathcal{F}}^{p,T}(v_\varepsilon^l)$ tends to $d_q\psi^u(T,p)(d_q\psi^u(t_l,p))^{-1}Y_l = \tilde{Z}_l$ as ε tends to zero. Since the the range of $D_u E_{\mathcal{F}}^{p,T}$ is closed, we deduce that it contains $\tilde{Z}_l$ for every $1 \leq l \leq d$.

We have therefore proved that u is o-regular at p in time T in the sense of Definition 6.3.3.

Remark 6.3.9. In contrast to what was discussed in Remark 6.3.8, we highlight the fact that in general case where the (finite) family $\mathcal{F}$ of vector fields does not satisfy (everywhere on the orbit) the Hörmander condition $\mathrm{Lie}_q\mathcal{F} = T_q\mathcal{O}_{\mathcal{F}}(p)$, for a given control $u_0 \in U_{\mathcal{F}}^{p,T}$ the o-regular controls u (in the sense of Definition 6.3.5) such that $E_{\mathcal{F}}^{p,T}(u_0) = E_{\mathcal{F}}^{p,T}(u)$ might lie far away from u_0 in L^2-sense.

As the standard example, consider on $M = \mathbb{R}^2$, with coordinates (x,y), the vector fields (cf. [17], p.12) $X = \frac{\partial}{\partial x}$ and $Y = \phi(x)\frac{\partial}{\partial y}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is smooth such that $\phi(x) = 0$ if $x \leq 0$ and $\phi(x) > 0$ for $x > 0$. Let $\mathcal{F} = \{X, Y\}$ and $\mathbb{R}^2_- = \{(x,y) \mid x < 0\}$

It is clear that for any point $p_0 = (x_0, y_0)$ with $x_0 < 0$, any $T > 0$ and any control u_0 such that $E_{\mathcal{F}}^{p_0,t}(u_0) \in \mathbb{R}^2_-$ for all $t \in [0,T]$, there is an L^2-neighbourhood of u_0 such that $E_{\mathcal{F}}^{p_0,T}$ is not regular at any of its points.

A regular control u steering p_0 to $q_0 = E_{\mathcal{F}}^{p_0,T}(u_0)$ in time T (i.e. $E_{\mathcal{F}}^{p_0,T}(u) = q$), which exists thanks to Proposition 6.3.7, must have the property that $E_{\mathcal{F}}^{p_0,T_0}(u) \notin \mathbb{R}^2_-$ for some $0 < T_0 \leq T$. Therefore, if we write $\gamma_u(t) = (x_u(t), y_u(t)) = E^{p_0,t}(u)$ and $u = (u_1, u_2)$, one has

$$|x_0| \leq |x_u(T_0) - x_0| = \Big| \int_0^{T_0} u_1(s)ds \Big| \leq \sqrt{T_0}\, \|u_1\|_{L^2([0,T])} \leq \sqrt{T}\, \|u\|_{L^2([0,T],\mathbb{R}^2)} .$$

If for example one took $u_0 = 0$, hence $q_0 = p_0$, the above inequality would prove, as was claimed above, that a regular control u steering p_0 to q_0 in time T cannot be near u_0 in L^2-sense.

6.3.2 Connectedness of $\mathcal{H}_{\Delta_R}^{\nabla}$ and H_{Δ}^{∇}

To prove that these groups are connected, it is enough to show that, for every $g \in \mathbb{G}_m$, the set $\Omega_\Delta(g)$ of all a.c. Δ-admissible loops based at $g \in \mathbb{G}_m$ is itself connected.

For that purpose, we introduce some notations. For real-valued measurable functions u, v defined on $[0,1]$, define $U(t) = \int_0^t u(s)ds$ and $u * v(t) = \int_0^t V(s)u(s) - U(s)v(s)ds$ for $t \in [0,1]$. Set $H_0 = (L_0^2)^m$, where L_0^2 as the subspace of functions $u \in L^2([0,1],\mathbb{R})$ with zero mean, i.e., $U(1) = 0$. Note then that for every $u \in L_0^2$ and $v \in L^2([0,1],\mathbb{R})$, one has $u * v(1) = -2\langle v, U\rangle$ where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2([0,1],\mathbb{R})$. One deduces that H_0 is a subspace of H of codimension m. Finally, for $u \in L_0^2$, let $L(u)$ be the subspace of L_0^2 of functions v such that $u * v(1) = 0$. This is a hyperplane in L_0^2 and therefore it is trivially convex. Moreover the relation "$v \in L(u)$" is symmetric because $u * v$ is a skew-symmetric operation.

Fix $\bar{g} \in \mathbb{G}_m$. Consider now the set of a.c. Δ-admissible curves starting at $\bar{g}$, i.e., the solutions of the Cauchy problems $\dot{g} = \sum_{1 \leq i \leq m} u_i X_i(g)$, $g(0) = \bar{g}$, where $\mathfrak{u} =$

$(u_1, \dots, u_m) \in H := L^2([0,1], \mathbb{R}^m)$. In coordinates $g = (x, \gamma)$ with $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $\gamma = (\gamma_{hk})_{(h,k)\in\mathcal{I}}$, one has

$$\begin{aligned} \dot{x}_i &= u_i, && \text{for } i = 1, \dots, m; \\ \dot{\gamma}_{hk} &= \frac{1}{2}(x_k u_h - x_h u_k), && \text{for } (h,k) \in \mathcal{I}. \end{aligned}$$

The terminal point $g(1) = (x(1), \gamma(1))$ is then equal to $\bar{g} + \big((U_i(1))_{1\le i\le m}, (\frac{1}{2}(u_k * u_h)(1))_{(h,k)\in\mathcal{I}}\big)$.

Define Loop_m as the subset functions $\mathfrak{u} \in H_0$ such that $(u_k * u_h)(1) = 0$ for $(h,k) \in \mathcal{I}$, i.e. $u_l \in L(u_h)$ (or equivalently $u_h \in L(u_l)$) for $(h,k) \in \mathcal{I}$. One derives at once that $\Omega_\Delta(\bar{g})$ is the set of terminal points for a.c. Δ-admissible curves starting at $\bar{g}$ corresponding to control functions $\mathfrak{u} \in \mathrm{Loop}_m$. The conclusion holds true if one shows that Loop_m itself is path-connected.

To see that, consider $\mathfrak{u} = (u_1, ..., u_m)$ in Loop_m and notice that, for every $\lambda \in \mathbb{R}$, then $\lambda\mathfrak{u}$ also belongs to Loop_m. Therefore the latter is contractible, implying in particular that it is connected. Hence the conclusion.

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